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Algorithmic and logical characterizations of bisimulations for non-deterministic fuzzy transition systems

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Abstract

Bisimulation is a well-known behavioral equivalence for discrete event systems and has been developed in fuzzy systems quickly. In this paper, we adopt an approach of the relational lifting that is one of the most used methods in the field of bisimulation research, to define it for a non-deterministic fuzzy transition system. An $\mathcal{O}(|S|^4|{\rightarrow}|^2)$ algorithm is given for testing bisimulation where $|S|$ is the number of states and $|{\rightarrow}|$ the number of transitions in the underlying transition systems. Two different modal logics are presented. One is two-valued and indicates whether a state satisfies a formula, which is an extension of Hennessy–Milner logic. The other is real-valued and shows to what extent a state satisfies a formula. They both characterize bisimilarity soundly and completely. Interestingly, the second characterization holds under a class of fuzzy logics. In addition, this real-valued logic allows us to conveniently define a logical metric that captures the similarity between states or systems. That is, the smaller distance, the more states alike. Although the work is inspired by the corresponding work in probabilistic systems, it is obviously different. In particular, the real-valued logic in this paper remains unexplored in fuzzy systems, even in probabilistic systems.

Keywords: Bisimulation; Fuzzy transition system; Modal logic; Logical characterization; Logical metric

1. Introduction

Bisimulations are established forms of behavioral equivalences for discrete event systems like process algebras, Petri nets or automata models and have been widely used in many areas of computer science. They are helpful to model-checking by reducing the number of states of systems.

Bisimulations have attracted much attention from researchers who work in the field of fuzzy systems and have been developed quickly [3,5–7,9,15,21,31,33]. The bisimulation of Cao et al. [5] is defined on an equivalence relation. This paper generalizes it to a general case. In this setting, we require that if (s, t) be a pair of states in a simulation relation

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R, i.e., sRt, then t can mimic all the stepwise behaviors of s with respect to R. Thus, if s can perform an action and evolve into a distribution μ, then t can perform the same action to another distribution ν such that ν can somehow simulate the behavior of μ according to R. To formalize the mimicking of μ by ν, we have to lift R to the relation $R^\dagger$ between distributions and require $\mu R^\dagger \nu$. We say that R is a bisimulation provided that R^{-1}, the inverse of R, is also a simulation.

A good scientific concept is often elegant, even seen from many different perspectives. Bisimulation is one of such concepts in classical and probabilistic systems, as it can be characterized in a great many ways such as fixed point theory, modal logics, game theory, coalgebras etc. We believe that (fuzzy) bisimulation is also one of such concepts in fuzzy systems. For example, Cao et al. [5] used the fixed point to characterize bisimulations, while Wu and Chen [32] characterized them by using coalgebras. We will provide in this paper two characterizations, from the perspectives of decision algorithms and modal logics.

The algorithm tests whether two states are bisimilar. It is realized through deciding whether they are in the greatest bisimulation (bisimilarity) that can be approached by a family inductively defined relations (see Definition 4.1). This algorithm is inspired by Baier in [1] but obviously different. Baier decided whether two (probability) distributions are related by some lifted relation in terms of the maximum flow algorithm, while we adopts a simple but subtle algorithm to do this for two (possibility) distributions (see Algorithm 1). The time complexity of the algorithm determining bisimulation is $\mathcal{O}(|S|^4|{\to}|^2)$ where $|S|$ is the number of states and $|{\to}|$ the number of transitions in the underlying transition systems.

Because of connections between modal logics and bisimulations, whenever a new bisimulation is proposed, the quest starts for the associated logic, such that two states or systems are bisimilar iff they satisfy the same modal logical formulae. Along this line, a great amount of work has appeared that characterizes various kinds of classical (or probabilistic) bisimulations by appropriate logics, e.g. [8,10,13,14,17,18,22,27,29,34]. Although, bisimulations have been investigated extensively in fuzzy systems, there is little work about the connections between (fuzzy) bisimulations and modal logics. Fan in [15] characterized (fuzzy) bisimulations for fuzzy Kripke structures in terms of Gödel modal logic; Wu and Deng in [31] characterized bisimilarity for (deterministic) fuzzy transition systems by using a fuzzy style Hennessy–Milner logic.

Another work of this paper is to characterize bisimilarity for (nondeterministic) fuzzy transition systems. We first give a Hennessy–Milner style modal logic. This logic is two-sorted and has state formulae and distribution formulae, which is different from the logic in [31] only including state formulae. It is two-valued in the sense whether a state satisfies a formula or not. We also provide a real-valued modal logic that shows to what extent a state satisfies a formula. To the best of our knowledge, this real-valued logic remains unexplored in the literature. Both these two logics characterize bisimilarity soundly and completely. Interestingly, the second characterization holds under a class of fuzzy logics. It should also be pointed out that two-valued logical characterization is motivated by the corresponding work in probabilistic systems. However, the proof in fuzzy case is more difficult than that in probabilistic case, see Theorem 5.2 and Remark 5.4 for details. Finally, with the help of the real-valued logic, we define a logical metric that is a more robust way of formalizing similarity between fuzzy systems than bisimulations. The smaller the logical distance, the more states behave similarly. In particular, the logical distance between two states is 0 iff they are exactly bisimilar.

The rest of this paper is structured as follows. We briefly review some basic concepts used in this paper in Section 2. Section 3 introduces the notions of lifting operation and bisimulation. Some properties about them are discussed. Moreover, an algorithm is given for determining whether two distributions are related by some lifted relation. In Section 4, we present an algorithm for testing bisimulation. In the subsequent section, we provide a two-valued and a real-valued logics, respectively. They both characterize bisimilarity soundly and completely. In Section 6, we define a logical metric to measure the similarity between states or systems. Finally, this paper is concluded in Section 7 with some future work.

2. Preliminaries

In this section, we briefly recall some notions used in this paper.

The notions about fuzzy set are mainly borrowed from [5]. Let S be a set and μ a *fuzzy set* in S. The *support* of μ is the set $\mathrm{supp}(\mu) = \{s \in S \mid \mu(s) > 0\}$. We denote by $\mathcal{F}(S)$ the set of all fuzzy sets in S. Whenever $\mathrm{supp}(\mu)$ is a finite set, say $\{s_1, s_2, \cdots, s_n\}$, then a fuzzy set μ can be written in Zadeh's notation as follows:

$$\mu = \frac{\mu(s_1)}{s_1} + \frac{\mu(s_2)}{s_2} + \cdots + \frac{\mu(s_n)}{s_n}.$$

With a slight abuse of notations, we sometimes write a possibility distribution to mean a fuzzy set.[1] For any $\mu \in \mathcal{F}(S)$ and $U \subseteq S$, the notation $\mu(U)$ stands for $\sup_{s\in U} \mu(s)$. In particular, $\mu(\emptyset) = 0$.

Given a binary relation $R \subseteq S \times S$, we sometimes write sRt if $(s, t) \in R$. An *equivalence relation* R partitions a set S into equivalence classes. For $s \in S$ we write $[s]_R$ to mean the unique equivalence class containing s. We drop the subscript R if the relation considered is clear from the context. For convenience, we let $\overrightarrow{R_s}$ and $\overleftarrow{R_s}$ denote the sets $\{s' \mid sRs'\}$ and $\{s' \mid s'Rs\}$, respectively. A set U is said to be R-*closed* if $\overrightarrow{R_s} \subseteq U$ for all $s \in U$.

The following lemma will be useful for the proof of Theorem 5.2 latter.

Lemma 2.1. *[31] Let $\mu, \nu \in \mathcal{F}(S)$ and R be an equivalence relation. If $\mu([s]) = \nu([s])$ for all R-equivalence classes $[s]$, then $\mu(U) = \nu(U)$ for any R-closed set U.*

3. Lifting and bisimulation

This section, consisting of two subsections, is devoted to the notions and some properties about relational lifting and bisimulation.

3.1. Relational lifting

In order to compare the behavior of two states, we often need to invoke relations between two distributions by lifting relations between states. Next, we will consider a typical lifting operation by using weight functions. It is originally given in [19] to compare probability distributions. A fuzzy analogue [5] is as follows.

Definition 3.1. Let $R \subseteq S \times S$ be a relation. The lifted relation $R^\dagger \subseteq \mathcal{F}(S) \times \mathcal{F}(S)$ is a relation over possibility distributions such that $\mu R^\dagger \nu$ iff a *weight function* $e : S \times S \to [0, 1]$ exists with respect to R and satisfies the following lifting conditions:

(E1) $\mu(s) = \sup_{t\in S} e(s, t)$, for any $s \in S$;
(E2) $\nu(t) = \sup_{s\in S} e(s, t)$, for any $t \in S$;
(E3) $e(s, t) = 0$, if $(s, t) \notin R$.

We sometimes write $e(s, U)$ (resp. $e(U, t)$) for $\sup_{t\in U} e(s, t)$ (resp. $\sup_{s\in U} e(s, t)$) for any $s \in S$ and $U \subseteq S$. In particular, $e(s, \emptyset) = 0$ and $e(\emptyset, s) = 0$ for any $s \in S$.

The following theorem gives a sufficient and necessary condition to determine whether two distributions are related by some lifted relation.

Theorem 3.2. *Let S be a set, $R \subseteq S \times S$ and $\mu, \nu \in \mathcal{F}(S)$. Then $\mu R^\dagger \nu$ iff for any $s \in S$,*

$$\mu(s) \leq \nu(\overrightarrow{R_s}) \text{ and } \nu(s) \leq \mu(\overleftarrow{R_s}).$$

Proof. "$\Longrightarrow$" Suppose $\mu R^\dagger \nu$. We first prove that $\mu(s) \leq \nu(\overrightarrow{R_s})$. Since $\mu R^\dagger \nu$ holds, a weight function e exists and satisfies E1, E2 and E3. Two cases need to be considered.

Case 1. If $\overrightarrow{R_s} = \emptyset$, then $\nu(\overrightarrow{R_s}) = 0$ and $\mu(s) = e(s, \overrightarrow{R_s}) = e(s, \emptyset) = 0$.

Case 2. If $\overrightarrow{R_s} \neq \emptyset$, then

[1] Strictly speaking a possibility distribution is different from a fuzzy set, though the former can be viewed as the generalized characteristic function of the latter. See [36] for more detailed discussion.

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$$\begin{aligned}
\nu(\overrightarrow{R_s}) &= \sup\nolimits_{t\in\overrightarrow{R_s}} \nu(t) \\
&= \sup\nolimits_{t\in\overrightarrow{R_s}} \sup\nolimits_{s'\in S} e(s',t) \\
&= \sup\nolimits_{s'\in S} \sup\nolimits_{t\in\overrightarrow{R_s}} e(s',t) \\
&= \sup\nolimits_{s'\in S} e(s',\overrightarrow{R_s}) \\
&\geq e(s,\overrightarrow{R_s}) \quad (\text{since } s\in S) \\
&= e(s,S) \\
&= \mu(s).
\end{aligned}$$

Hence, $\mu(s) \leq \nu(\overrightarrow{R_s})$ holds. In a similar way, we can prove that $\nu(s) \leq \mu(\overleftarrow{R_s})$.

"$\Longleftarrow$" Suppose that $\mu(s) \leq \nu(\overrightarrow{R_s})$ and $\nu(s) \leq \mu(\overleftarrow{R_s})$.

Let

$$W(s,t) = \begin{cases} \min(\mu(s),\nu(t)) & \text{if } (s,t)\in R \\ 0, & \text{otherwise.} \end{cases}$$

It suffices to verify that W satisfies E1 and E2. For any $s \in S$,

$$\begin{aligned}
\sup\nolimits_{t\in S} W(s,t) &= \sup\nolimits_{t\in\overrightarrow{R_s}} W(s,t) \\
&= \sup\nolimits_{t\in\overrightarrow{R_s}} \mu(s)\wedge\nu(t) \\
&= \mu(s)\wedge \sup\nolimits_{t\in\overrightarrow{R_s}} \nu(t) \\
&= \mu(s)\wedge\nu(\overrightarrow{R_s}) \\
&= \mu(s).
\end{aligned}$$

In a similar way, we can obtain that $\sup_{s\in S} W(s,t) = \nu(t)$. This completes the proof. □

This theorem shows that without help from weight functions, we can still determine whether $\mu R^\dagger \nu$ only depending on μ, ν and R themselves.

Based on Theorem 3.2, we give an algorithm to check if two distributions are in some lifted relation.

Algorithm 1 Given a finite set S, $R \subseteq S \times S$, μ, and ν, determine whether $\mu R^\dagger \nu$.

1: **function** $R^\dagger(S, R, \mu, \nu)$
2: Initialize arrays A and B to be zero.
3: **for all** $s \in S$ **do**
4: **if** $\mu(s) \leq \nu(\overrightarrow{R_s})$ **then**
5: $A[s] = 1$
6: **if** $\nu(s) \leq \mu(\overleftarrow{R_s})$ **then**
7: $B[s] = 1$
8: **if** $(\min_{s\in S}(A[s] \wedge B[s]) == 0)$ **then**
9: **return FALSE**
10: **return TRUE**

Clearly, Step 3–7 is $\mathcal{O}(|S|^2)$ and Step 8 is $\mathcal{O}(|S|)$. Hence, the Algorithm 1 is in $\mathcal{O}(|S|^2)$.

In the probabilistic setting, determine whether two (probability) distributions are related by some lifted relation adopts the maximum flow algorithm [1], while ours is completely different.

The operation $R^\dagger$ enjoys some interesting properties that are helpful for understanding Proposition 3.6 latter.

Lemma 3.3. *The operation $R^\dagger$ has the following properties.*

1. *It preserves the equality relation:*

$$(Eq(S))^\dagger = Eq(\mathcal{F}(S))$$

where $Eq(S) = \{(s,s) \mid s \in S\}$.

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2. *It is monotone:*

$$R_1 \subseteq R_2 \Longrightarrow R_1^\dagger \subseteq R_2^\dagger.$$

3. *It preserves relation composition:*

$$(R_1 \circ R_2)^\dagger = R_1^\dagger \circ R_2^\dagger$$

where $R_1 \circ R_2 = \{(s,t) \mid \exists s' \in S.\ s R_1 s' \wedge s' R_2 t\}$.

4. *It commutes with inverse of relation:*

$$(R^{-1})^\dagger = (R^\dagger)^{-1}.$$

5. *It preserves reflexivity, symmetry and transitivity, and thus, if R is an equivalence relation, then so is $R^\dagger$.*

Proof. As an example, we choose to prove 3. It suffices to prove that two sets $(R_1 \circ R_2)^\dagger$ and $R_1^\dagger \circ R_2^\dagger$ contain each other.

(1)

$$(R_1 \circ R_2)^\dagger \subseteq R_1^\dagger \circ R_2^\dagger.$$

Let $R_1, R_2 \subseteq S \times S$ and $(\mu, \eta) \in (R_1 \circ R_2)^\dagger$. Then, Theorem 3.2 implies that for any $s \in S$,

$$\mu(s) \le \eta(\overrightarrow{(R_1 \circ R_2)_s}) \text{ and } \eta(s) \le \mu(\overleftarrow{(R_1 \circ R_2)_s}). \qquad (\star)$$

For any $t \in S$, define:

$$\nu(t) = \sup\{\mu(s) \wedge \eta(s') \mid s \in \overleftarrow{(R_1)_t} \wedge s' \in \overrightarrow{(R_2)_t}\}.$$

Note that, if $\overleftarrow{(R_1)_t} = \emptyset$ or $\overrightarrow{(R_2)_t} = \emptyset$, then $\nu(t) = 0$.

Now, we prove that $(\nu, \eta) \in R_2^\dagger$. Again by Theorem 3.2, it suffices to verify that for any $t \in S$,

$$\nu(t) \le \eta(\overrightarrow{(R_2)_t}) \text{ and } \eta(t) \le \nu(\overleftarrow{(R_2)_t}).$$

On one hand,

$$\begin{aligned} \nu(t) &= \sup\{\mu(s) \wedge \eta(s') \mid s \in \overleftarrow{(R_1)_t} \wedge s' \in \overrightarrow{(R_2)_t}\} \\ &\le \sup\nolimits_{s' \in \overrightarrow{(R_2)_t}} \eta(s') \\ &= \eta(\overrightarrow{(R_2)_t}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \nu(\overleftarrow{(R_2)_t}) &= \sup\nolimits_{s \in \overleftarrow{(R_2)_t}} \nu(s) \\ &= \sup\nolimits_{s \in \overleftarrow{(R_2)_t}} \sup\{\mu(s_1) \wedge \eta(s_2) \mid s_1 \in \overleftarrow{(R_1)_s} \wedge s_2 \in \overrightarrow{(R_2)_s}\} \\ &\ge \sup\nolimits_{s \in \overleftarrow{(R_2)_t}} \sup\nolimits_{s_1 \in \overleftarrow{(R_1)_s}} \mu(s_1) \wedge \eta(t) \ (\text{since } t \in \overrightarrow{(R_2)_s} \text{ if } s \in \overleftarrow{(R_2)_t}) \\ &= \eta(t) \wedge \sup\{\mu(s_1) \mid s \in \overleftarrow{(R_2)_t} \wedge s_1 \in \overleftarrow{(R_1)_s}\} \\ &= \eta(t) \wedge \sup\nolimits_{s_1 \in \overleftarrow{(R_1 \circ R_2)_t}} \mu(s_1) \\ &= \eta(t) \wedge \mu(\overleftarrow{(R_1 \circ R_2)_t}) \\ &= \eta(t) \quad (\text{by } (\star)). \end{aligned}$$

Hence, $(\nu, \eta) \in R_2^\dagger$. In a similar way, we can also prove that $(\mu, \nu) \in R_1^\dagger$. It follows that $(\mu, \eta) \in R_1^\dagger \circ R_2^\dagger$ and then $(R_1 \circ R_2)^\dagger \subseteq R_1^\dagger \circ R_2^\dagger$ as desired.

(2)

$$R_1^\dagger \circ R_2^\dagger \subseteq (R_1 \circ R_2)^\dagger.$$

We directly use Definition 3.1 to prove it. Suppose that $(\mu, \eta) \in R_1^\dagger \circ R_2^\dagger$. Then there exists a ν such that $(\mu, \nu) \in R_1^\dagger$ and $(\nu, \eta) \in R_2^\dagger$. Further, two weight functions e_1 and e_2 exist and both satisfy E1, E2 and E3.

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Let

$$e(s,t) = \sup_{s' \in S} e_1(s,s') \wedge e_2(s',t).$$

It remains to verify that e satisfies E1, E2 and E3.

First, if $e(s,t) > 0$, then by the definition of e, $\exists s' \in S$ such that $e_1(s,s') > 0$ and $e_2(s',t) > 0$. It follows from e_1 and e_2 satisfying E3 that $(s,s') \in R_1$ and $(s',t) \in R_2$, implying $(s,t) \in R_1 \circ R_2$. That is, e satisfies E3.

Second,

$$\begin{aligned} e(s,S) &= \sup_{t\in S} e(s,t) \\ &= \sup_{t\in S} \sup_{s'\in S} e_1(s,s') \wedge e_2(s',t) \\ &= \sup_{s'\in S} \sup_{t\in S} e_1(s,s') \wedge e_2(s',t) \\ &= \sup_{s'\in S} [e_1(s,s') \wedge \sup_{t\in S} e_2(s',t)] \\ &= \sup_{s'\in S} e_1(s,s') \wedge \nu(s') \quad \text{(since } e_2 \text{ satisfies E1)} \\ &= \sup_{s'\in S} e_1(s,s') \\ &= \mu(s) \quad \text{(since } e_1 \text{ satisfies E1).} \end{aligned}$$

The second step from the bottom holds since e_1 satisfies E2 and then $\nu(s') \geq e_1(s,s')$ for any $s \in S$. In a similar way, we can also get that $e(S,t) = \eta(t)$. In summary, e satisfies E1, E2 and E3 and hence $(\mu,\eta) \in (R_1 \circ R_2)^\dagger$. So, $R_1^\dagger \circ R_2^\dagger \subseteq (R_1 \circ R_2)^\dagger$ holds. This completes the proof. □

3.2. Bisimulation

In this subsection, we introduce bisimulation. First, we recall the notion of the nondeterministic fuzzy transition system (**NFTS**, for short).

Definition 3.4. [5] An **NFTS** is a triple $\mathcal{M} = (S, A, \rightarrow)$, where S is a set of states, A is a set of actions, and $\rightarrow \subseteq S \times A \times \mathcal{F}(S)$ is the transition relation.

From each state s, more than one possibility distribution may be reached by performing action a, while in an **FTS** (deterministic fuzzy transition system), at most one possibility distribution may be reached. We sometimes write $s \xrightarrow{a} \mu$ for $(s,a,\mu) \in \rightarrow$. For each $a \in A$ and $s \in S$, we let $T_a(s) = \{\mu \in \mathcal{F}(S) \mid s \xrightarrow{a} \mu\}$. An **NFTS** is *image-finite* if for each $a \in A$ and $s \in S$, the set $T_a(s)$ is finite and for any $\mu \in T_a(s)$, the $\mathrm{supp}(\mu)$ is also finite; an **NFTS** is *finitely branching* if for each $s \in S$, the set $\{(a,\mu) \mid s \xrightarrow{a} \mu, a \in A, \mu \in \mathcal{F}(S)\}$ is finite, further, this **NFTS** is *finitary* if the set S is also finite.

In the following, we motivate our work from two aspects: differences between an **FTS** and an **NFTS**, and some limitations of fuzzy language equivalence.

To better understand them, we consider an example related to medical diagnosis and treatment, as described by Cao et al. [5], Qiu [28] and Xing et al. [33]. We assume that there is an unknown bacterial infection. Based on his experience, a physician believes that two drugs, say a_1 and a_2, may be useful for treating this disease. Three possible negative symptoms, e.g., b_1, b_2, b_3, must also be considered. A patient's condition can be in one of four rough types, e.g., "poor", "fair", "good", and "excellent", which are denoted by the capital letters P, F, G and E, respectively. A treatment (or a negative symptom) may lead to a state among multiple possible states with certain degree. For example, the transition $F \xrightarrow{a_1[0.6]} E$ means that the patient's condition has changed from "fair" to "excellent" with possibility 0.6 after using drug a_1, whereas $F \xrightarrow{b_1[0.3]} P$ means that the patient's condition has changed from "fair" to "poor" with possibility 0.3 if the patient has negative symptom b_1. The transition possibilities of these events are estimated by the physician. Different physical conditions of patients may lead to nondeterministic changes even if the patients are in the same state and are given the same treatment. For example, we may have the following two transitions:

$$P \xrightarrow{a_1} \{\frac{0.2}{P} + \frac{0.7}{F}\} \qquad \text{and} \qquad P \xrightarrow{a_1} \{\frac{0.3}{F} + \frac{0.6}{G} + \frac{0.2}{E}\}.$$

In this manner, we obtain an **NFTS** $(S, A, \rightarrow)$, where $S = \{P, F, G, E\}$, $A = \{a_1, a_2, b_1, b_2, b_3\}$ and $\rightarrow \subseteq S \times A \times \mathcal{F}(S)$.

A patient's initial condition may be "poor" and it will become "fair," "good," or even "excellent" after a specific treatment. When a patient's health becomes "fair," we naturally hope that this will improve to become "excellent" instead of deteriorating. Analogously, if the patient's condition is "excellent," it is desirable to maintain good health and thus a supervisor is necessary to disable events b_1, b_2 and b_3 if they are controllable. A general approach for determining whether supervisory control exists for fuzzy discrete event systems is in terms of fuzzy language equivalence. However, this is not satisfactory because some strings (negative symptoms) are not accepted if they are controllable. Xing et al. [33] proposed the use of fuzzy bisimulation equivalence to solve this problem, while Deng and Qiu [11] also noted some limitations of fuzzy language equivalence. They addressed this problem using fuzzy simulation equivalence whose expressiveness is stronger than that of fuzzy language equivalence.

Cao et al. [5] studied the behavioral distance for **NFTS**s. It is a more robust way of formalizing behavioral similarity than bisimulations, which can be applied to quantitative verification of systems. Cao and Ezawa [4] pointed out that deterministic fuzzy automata and nondeterministic fuzzy automata that extend **FTS**s and **NFTS**s with an initial state and a fuzzy set over final states, respectively, accept the same class of fuzzy languages. That is, for any nondeterministic fuzzy automata $\mathcal{M}$, there exists a deterministic fuzzy automata $\mathcal{M}'$ such that they are (weakly) language-equivalent. Recently, Pan et al. [26] investigated strong language equivalence. They pointed out that strong language equivalence is strictly finer than weak language equivalence and the expressive power of nondeterministic fuzzy automata is more powerful than that of deterministic fuzzy automata with respect to strong language equivalence. Please see [4,26] for more details.

In addition, Cao and Ezawa noted that an **NFTS** can easily be combined into an **FTS**. However, after their combination, the transitions in the deterministic system can differ from any transition in the nondeterministic system. For instance, by combining the transitions above and taking their supremum, we obtain the following transition:

$$P \xrightarrow{a_1} \{\frac{0.2}{P} + \frac{0.7}{F} + \frac{0.6}{G} + \frac{0.2}{E}\},$$

which is completely different from any one of the two transitions above. From a semantic view point, we consider that this way of reducing **NFTS**s to **FTS**s is unsatisfactory. Moreover, the expressive power of the logics characterizing bisimilarity, respectively, for **FTS**s and **NFTS**s, are also different (see Section 5), which in turn shows the difference between these two systems.

Based on these reasons, we consider bisimulations for **NFTS**s. Main contributions include algorithmic and logical characterizations that are useful for model checking. We are now ready to introduce the notion of bisimulation.

Definition 3.5. A relation $R \subseteq S \times S$ is a *simulation* if whenever sRt, then for any transition $s \xrightarrow{a} \mu$, there exists a transition $t \xrightarrow{a} \nu$ such that $\mu R^{\dagger} \nu$. If both R and R^{-1} are simulations, then R is a*bisimulation*. Two states $s, t \in S$ are *bisimilar*, denoted as $s \sim t$, if there is a bisimulation R with sRt.

If R is an equivalence relation, then Definition 3.5 coincides with Definition 9 of Cao et al. [5]. That is, an equivalence relation $R \subseteq S \times S$ is a bisimulation if whenever sRt, then for any transition $s \xrightarrow{a} \mu$, there exists a transition $t \xrightarrow{a} \nu$ such that $\mu([s']) = \nu([s'])$ for all R-equivalence classes $[s']$.

Bisimulation is preserved by the equality, inverse, union and the composition of relations, which are summarized in the following proposition.

Proposition 3.6. *The following statements hold.*

1. *$Eq(S)$ is a bisimulation;*
2. *If R is a bisimulation, then so is R^{-1};*
3. *If R_i is a bisimulations $(i \in I)$, then so is $\bigcup_{i \in I} R_i$;*
4. *If R_i is a bisimulation $(i = 1, 2)$, then so is $R_1 \circ R_2$.*

Proof. The proofs are not hard by Lemma 3.3 and hence are omitted. □

Proposition 3.6 (3) implies that $\sim$ is the greatest bisimulation because it is the union of all bisimulations, called *bisimilarity*. Moreover, it is an equivalence relation by Proposition 3.6 (1, 2, 4). Hence, if $s \sim t$, then for any transition

$s \xrightarrow{a} \mu$, there exists a transition $t \xrightarrow{a} \nu$ such that $\mu([s']) = \nu([s'])$ for all $\sim$-equivalence classes $[s']$. This fact will be used in the proof of Theorem 5.2.

The following lemma will be useful for analyzing Example 5.1 latter.

Lemma 3.7. *Let μ, ν be two distributions over S and R a symmetric relation. If $\mu R^\dagger \nu$ then $\mu(U) = \nu(U)$ for any R-closed set U.*

Proof. Suppose that $\mu R^\dagger \nu$. Then a weight function e exists and satisfies E1, E2 and E3. For any R-closed set U, we have

$$\begin{aligned}
\mu(U) &= \sup_{s\in U} \mu(s) \\
&= \sup_{s\in U} \sup_{t\in S} e(s,t) \\
&= \sup_{\{(s,\ t)\in R | s\in U\}} e(s,t) \\
&= \sup_{\{(s,\ t)\in R | t\in U\}} e(s,t) \\
&= \sup_{t\in U} \sup_{s\in S} e(s,t) \\
&= \sup_{t\in U} \nu(t) \\
&= \nu(U)
\end{aligned}$$

The fourth equality above needs a bit explanation. On one hand, if $s \in U$ and $(s, t) \in R$ then $t \in U$ because U is R-closed. On the other hand, by the symmetry of R we see from $(s, t) \in R$ that $(t, s) \in R$. Then if $t \in U$ we obtain $s \in U$ because U is R-closed. As a result, we have that $\{(s, t) \in R \mid s \in U\} = \{(s, t) \in R \mid t \in U\}$. $\square$

The converse of this lemma does not hold, which is witnessed by the following example.

Example 3.8. Let $S = \{s_1, s_2\}$ and $R = \{(s_1, s_2), (s_2, s_1)\}$. Consider the following two distributions:

$$\mu = \frac{0.2}{s_1} + \frac{0.3}{s_2}, \qquad \nu = \frac{0.1}{s_1} + \frac{0.3}{s_2}.$$

The only non-empty R-closed set is S and $\mu(S) = \nu(S) = 0.3$. However, Theorem 3.2 implies that $(\mu, \nu) \notin R^\dagger$ because $\overrightarrow{R_{s_2}} = \{s_1\}$ and $\mu(s_2) > \nu(\overrightarrow{R_{s_2}})$.

The bisimulation for an **FTS** given by Wu and Deng in [31] can be generalized to an **NFTS**. That is, a relation $R \subseteq S \times S$ is a bisimulation if whenever sRt, then for any transition $s \xrightarrow{a} \mu$, there exists a transition $t \xrightarrow{a} \nu$ such that $\mu(U) = \nu(U)$ for any R-closed set U. Lemma 3.7 and Example 3.8 show that the bisimulation in Definition 3.5 is finer than this under the symmetric assumption. But, when restricted to an equivalence relation, these two notions are equivalent.

4. Algorithmic characterization

In this section, we present an algorithm for checking if two states s and t are bisimilar in a finitary **NFTS**. The most direct method to do it is to judge whether the pair (s, t) is in the bisimilarity. For this purpose, we need a little preparation.

As in the classical setting, (fuzzy) bisimilarity can be approached by a family inductively defined relations in a finitary **NFTS**.

Definition 4.1. Let $(S, A, \rightarrow)$ be an **NFTS**. We define:

- $R_0 := S \times S$;
- $sR_{n+1}t$, for $n \geq 0$, if
  1. whenever $s \xrightarrow{a} \mu$, then a transition $t \xrightarrow{a} \nu$ exists such that $\mu R_n^\dagger \nu$;
  2. whenever $t \xrightarrow{a} \nu$, then a transition $s \xrightarrow{a} \mu$ exists such that $\nu R_n^\dagger \mu$.

Lemma 4.2. *Let $(S, A, \to)$ be an* **NFTS**. *Then*

(1) R_n is symmetric for all $n \geq 0$;
(2) $R_n \supseteq R_{n+1}$ for all $n \geq 0$;
(3) If $\mathcal{M} = (S, A, \to)$ is finitely branching, then $\sim = \bigcap_{n\geq 0} R_n$.

Proof. (1) is straightforward. Making use of Lemma 3.3 (2), we can prove (2) by induction, we omit it. (3) can be similar to prove by Proposition 4.4 in [13], we also omit it. □

Now, we present an algorithm for testing bisimulation.

Algorithm 2 Given a finitary transition system $(S, A, \to)$ and $s_0, t_0 \in S$, determine whether $s_0 \sim t_0$.

1: **function** Bis(s_0, t_0)
2: Let $R_0 = S \times S$, $j = 0$.
3: **repeat**
4: $j := j + 1$
5: $R_j := R_{j-1}$
6: **for all** $(s, t) \in R_{j-1}$ **do**
7: **for all** $s \xrightarrow{a} \mu$ **do**
8: **if** there does not exist $t \xrightarrow{a} \nu$ such that $\mu R^{\dagger}_{j-1} \nu$ **then**
9: $R_j := R_j \backslash \{(s, t), (t, s)\}$
10: **until** $R_j = R_{j-1}$
11: **if** $(s_0, t_0) \notin R_j$ **then**
12: **return FALSE**
13: **return TRUE**

Proposition 4.3. *The Algorithm 2 correctly determine whether s_0 and t_0 are bisimilar and is in $\mathcal{O}(|S|^4|\to|^2)$.*

Proof. Definition 4.1 and Lemma 4.2 show that in a finitary transition system, if we can find a j such that $R_j = R_{j-1}$, then $\sim = R_j$. Hence, the algorithm correctly answers whether s_0 and t_0 are bisimilar when it terminates.

As complexity, Step 5–9 equal to determine for all rules like $s \xrightarrow{a} \mu$, whether there exists a rule $t \xrightarrow{b} \nu$ satisfying $(s, t) \in R_{j-1}$, $a = b$ and $\mu R^{\dagger}_{j-1} \nu$. Since checking whether $\mu R^{\dagger}_{j-1} \nu$ can be done by Algorithm 1 in $\mathcal{O}(|S|^2)$, Step 5–9 take $\mathcal{O}(|S|^2|\to|^2)$. In order to find a j such that $R_j = R_{j-1}$, the algorithm loops at most $\frac{|S|^2-|S|}{2}$ times. This is because that (s_i, s_i) $(i = 1, 2, \cdots, |S|)$ cannot be deleted from any R_j, as $s_i \sim s_i$. On the other hand, if (s_i, s_j) $(i \neq j)$ is deleted from some R_m, then (s_j, s_i) is also deleted from this R_m because R_m is symmetric. That is, if $R_j \neq R_{j-1}$, then $|R_j| \leq |R_{j-1}| - 2$. Hence, the time complexity of this algorithm is $\mathcal{O}(|S|^4|\to|^2)$. □

5. Logical characterizations

In this section, we embark upon the relationship between bisimilarity and logics. More concretely, we will show that two states are bisimilar iff they satisfy the same logical formulae or they have the same values on logical formulae.

5.1. Two-valued logic

In [31], Wu and Deng gave the logic $\mathcal{L}^{bi}$ as follows:

$$\varphi ::= \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a \rangle_p \varphi,$$

where $a \in A$ and $p \in [0, 1]$. This is the basic logic that we employ to establish the logical characterization of bisimilarity for an **FTS**. But, as we see in the next example that is adapted from Example 5.1 [10], it is not enough for the more general setting of **NFTS**s. Here, in an **NFTS**, the formulae in the logic $\mathcal{L}^{bi}$ are interpreted as usual. For example, $s \models \langle a \rangle_p \varphi$ means that there exists a transition $s \xrightarrow{a} \mu$ such that $\mu([\![\varphi]\!]) \geq p$.

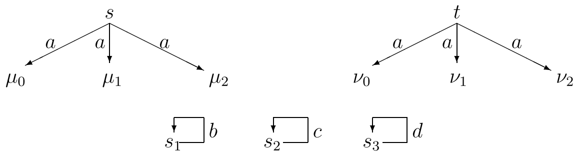

Fig. 1. s and t are not bisimilar but cannot be distinguished by $\mathcal{L}^{bi}$.

	μ_0	μ_1	μ_2	ν_0	ν_1	ν_2
$\{s_1\}$	0.2	0.6	0	0.6	0.2	0
$\{s_2\}$	0	0.2	0.6	0	0.6	0.2
$\{s_3\}$	0.6	0	0.2	0.2	0	0.6

Example 5.1. Consider the **NFTS** shown in the Fig. 1 where μ_0, μ_1, μ_2 and ν_0, ν_1, ν_2 are given in the diagram above. States s and t are not bisimilar. Suppose for a contradiction that they are bisimilar. Then a bisimulation R exists such that $(s, t) \in R$. Let $R' = R \cup R^{-1}$. Then R' is a symmetric relation and Proposition 3.6 (2, 3) implies that it is also a bisimulation. It follows from Lemma 3.7 that for a given $\mu \in T_a(s)$, a $\nu \in T_a(t)$ exists such that $\mu(U) = \nu(U)$ for all R'-closed sets U. However, this does not hold since the only relevant possible R'-closed subsets $U \in \{\{s_1\}, \{s_2\}, \{s_3\}\}$. Hence s and t are not bisimilar. The logic $\mathcal{L}^{bi}$ having a modality that can only describe one behavior after a label will not be able to distinguish between s and t. For example, for any $p \in [0, 1]$ and any formula $\varphi \in \mathcal{L}^{bi}$, either s and t both satisfy $\langle a\rangle_p\varphi$, or both not satisfy $\langle a\rangle_p\varphi$. Moreover, the negation, conjunction do not add any distinguishing power.

Hence the logic $\mathcal{L}^{bi}$ cannot characterize bisimilarity for **NFTS**s (after the proof of Theorem 5.2, we continue explaining the reason). We need a new logic that originates from Jonsson et al. [20]. This logic is two-sorted and has two kinds of formulae: one that is interpreted on states and another that is interpreted on possibility distributions.

Let A be a set of *actions* ranged over $a, b, \cdots$, and let $\top$ be a propositional constant. The language $\mathcal{L}$ of formulae is given by the following BNF grammar:

$$\begin{aligned} \varphi &::= \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a\rangle\psi \\ \psi &::= \psi_1 \wedge \psi_2 \mid \neg\psi \mid [\varphi]_p \end{aligned}$$

where $a \in A$ and $p \in [0, 1]$. We call φ a *state formula* and ψ a *distributive formula*.

Fix an **NFTS** $(S, A, \to)$, the *satisfaction relation* is defined by

- $s \models_{\mathcal{L}} \top$, for any $s \in S$;
- $s \models_{\mathcal{L}} \varphi_1 \wedge \varphi_2$ iff $s \models \varphi_i$ for $i = 1, 2$;
- $s \models_{\mathcal{L}} \neg\varphi$ iff $s \not\models_{\mathcal{L}} \varphi$;
- $s \models_{\mathcal{L}} \langle a\rangle\psi$ iff a transition $s \xrightarrow{a} \mu$ exists such that $\mu \models_{\mathcal{L}} \psi$;
- $\mu \models_{\mathcal{L}} \psi_1 \wedge \psi_2$ iff $\mu \models_{\mathcal{L}} \psi_i$ for $i = 1, 2$;
- $\mu \models_{\mathcal{L}} \neg\psi$ iff $\mu \not\models_{\mathcal{L}} \psi$;
- $\mu \models_{\mathcal{L}} [\varphi]_p$ iff $\mu(\{s \mid s \models_{\mathcal{L}} \varphi\}) \geq p$.

Our logic $\mathcal{L}$ differs from the logic [20] in two aspects: one is that distribution formulae [20] include $\top$, the other is that distribution formulae [20] are interpreted over probability distributions.

Let $[\![\varphi]\!]_{\mathcal{L}} = \{s \mid s \models_{\mathcal{L}} \varphi\}$, and let $Th_{\mathcal{L}}(s) = \{\varphi \in \mathcal{L} \mid s \models \varphi\}$ be the set of state formulae that are satisfied by s, i.e., the theory of s.

In the rest of this subsection, we prove the logical characterization theorem, which says that two states are bisimilar iff they are logically indistinguishable.

Theorem 5.2. *Let the set S be countably infinite and the* **NFTS** $(S, A, \to)$ *be image-finite. Then for any two states $s, t \in S$, $s \sim t$ iff $Th_{\mathcal{L}}(s) = Th_{\mathcal{L}}(t)$.*

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Proof. ($\Longrightarrow$) Let $s, t \in S$ be any two states and $s \sim t$. We show that $s \models_{\mathcal{L}} \varphi \Leftrightarrow t \models_{\mathcal{L}} \varphi$. Since $\sim$ is an equivalence relation and hence is symmetric, we only need to prove $s \models_{\mathcal{L}} \varphi \Rightarrow t \models_{\mathcal{L}} \varphi$. We proceed it by structural induction on $\varphi \in \mathcal{L}$. The cases of $\top$, conjunction and negation are trivial. Now consider the case $\varphi \equiv \langle a \rangle \psi$. Let $s \models_{\mathcal{L}} \langle a \rangle \psi$. There are three possibilities:

- If $\psi \equiv \psi_1 \wedge \psi_2$, then $s \models_{\mathcal{L}} \langle a \rangle (\psi_1 \wedge \psi_2)$. So, a transition $s \xrightarrow{a} \mu$ exists such that $\mu \models_{\mathcal{L}} \psi_1 \wedge \psi_2$, i.e., $\mu \models_{\mathcal{L}} \psi_i$ for $i = 1, 2$. Further since $s \sim t$, a transition $t \xrightarrow{a} \nu$ exists such that $\mu (\sim)^{\dagger} \nu$. By induction, we have $\nu \models_{\mathcal{L}} \psi_i$ for $i = 1, 2$, and then $\nu \models_{\mathcal{L}} \psi_1 \wedge \psi_2$. It follows that $t \models_{\mathcal{L}} \langle a \rangle (\psi_1 \wedge \psi_2)$.
- If $\psi \equiv \neg \psi_1$, the analysis is trivial.
- If $\psi \equiv [\phi]_p$, then $s \models_{\mathcal{L}} \langle a \rangle [\phi]_p$. So, a transition $s \xrightarrow{a} \mu$ exists such that $\mu \models_{\mathcal{L}} [\phi]_p$. That is, $\mu([\![\phi]\!]_{\mathcal{L}}) \geq p$. Since $s \sim t$, a transition $t \xrightarrow{a} \nu$ exists such that $\mu (\sim)^{\dagger} \nu$. By induction, $[\![\phi]\!]_{\mathcal{L}}$ is $\sim$-closed, Lemma 2.1 implies that $\nu([\![\phi]\!]_{\mathcal{L}}) = \mu([\![\phi]\!]_{\mathcal{L}}) \geq p$, i.e., $\nu \models_{\mathcal{L}} [\phi]_p$. Hence $t \models_{\mathcal{L}} \langle a \rangle [\phi]_p$.

($\Longleftarrow$) Define $R = \{(s, t) \mid Th_{\mathcal{L}}(s) = Th_{\mathcal{L}}(t)\}$. Then R is obvious an equivalence relation. It suffices to prove that R is a bisimulation.

Let $G = \{[s_i] \mid i \in I\}$ be the set of all equivalence classes of R. Note that G is countable because of our restriction to countable state space S. For all $i, j \in I$, if $s_j \notin [s_i]$, that is, $j \neq i$, then $Th_{\mathcal{L}}(s_i) \neq Th_{\mathcal{L}}(s_j)$, thus since $\mathcal{L}$ is closed under negation, a formula φ_{ij} exists such that $s_i \models_{\mathcal{L}} \varphi_{ij}$ but $s_j \not\models_{\mathcal{L}} \varphi_{ij}$. We fix an arbitrary index k. For each $i \in I$, define $\Phi_i^k = \bigwedge_{j \leq k} \varphi_{ij}$. It is then easy to see that for each $i \in I$, the formula Φ_i^k only has finite conjunctions, and

$$[s_i] \subseteq [\![\Phi_i^k]\!]_{\mathcal{L}} \subseteq [s_i] \cup \bigcup_{m \in I \wedge m > k} [s_m]. \tag{1}$$

For any $i \in I$ and any $\mu \in \mathcal{F}(S)$ with finite support, it is easy to get that

$$\mu([s_i]) = \inf_{k \in I} \mu([\![\Phi_i^k]\!]_{\mathcal{L}}). \tag{2}$$

Now let $(s, t) \in R$, and $s \xrightarrow{a} \mu$. It remains to show that there exists a transition $t \xrightarrow{a} \nu$ such that $\mu([s_i]) = \nu([s_i])$ $(i \in I)$.

Define

$$\Phi^k = \bigwedge_{i \in I \wedge i \leq k} [\Phi_i^k]_{p_i}$$

where $p_i = \mu([s_i])$. Obviously, Φ^k only has finite conjunctions for arbitrary k. The left part in Equation (1) shows that $\mu([\![\Phi_i^k]\!]_{\mathcal{L}}) \geq p_i = \mu([s_i])$ for all $i \leq k$, implying $\mu \models_{\mathcal{L}} \Phi^k$. Since k is arbitrary, $\mu \models_{\mathcal{L}} \Phi^k$ for all $k \in I$.

On the other hand, fix an $\epsilon > 0$, by the Equation (2), $\forall i \in I$, $\exists l_{i,\epsilon} \in I$ and a formula $\Phi_i^{l_{i,\epsilon}}$ such that

$$\mu([\![\Phi_i^{l_{i,\epsilon}}]\!]_{\mathcal{L}}) < \mu([s_i]) + \epsilon.$$

Now, let us fix a k and let $l = \max_{i \leq k} l_{i,\epsilon}$. Then by the construction of Φ_i^k, we have that $[\![\Phi_i^l]\!]_{\mathcal{L}} \subseteq [\![\Phi_i^{l_{i,\epsilon}}]\!]_{\mathcal{L}}$ for any $i \leq k$. It follows that for any $i \leq k$,

$$\mu([\![\Phi_i^l]\!]_{\mathcal{L}}) \leq \mu([\![\Phi_i^{l_{i,\epsilon}}]\!]_{\mathcal{L}}) < \mu([s_i]) + \epsilon = p_i + \epsilon.$$

Define

$$\Psi^k = \bigwedge_{i \in I \wedge i \leq k} \neg [\Phi_i^l]_{p_i + \epsilon}.$$

Then Ψ^k only has finite conjunctions and $\mu \models_{\mathcal{L}} \Psi^k$. Since k can be arbitrary, $\mu \models_{\mathcal{L}} \Psi^k$ for all $k \in I$.

Hence $s \models_{\mathcal{L}} \langle a \rangle (\Phi^k \wedge \Psi^k)$ for all $k \in I$. By the definition of R, $t \models_{\mathcal{L}} \langle a \rangle (\Phi^k \wedge \Psi^k)$ for all $k \in I$. From state t there are only finitely many outgoing transitions labeled by a since we are considering an image-finite **NFTS**, thus the Pigeonhole principle applies and a ν_ϵ exists such that $t \xrightarrow{a} \nu_\epsilon$ and $\nu_\epsilon \models_{\mathcal{L}} \Phi^k \wedge \Psi^k$ for infinitely many indices $k \in \mathbb{Z}$ with $\mathbb{Z} \subseteq I$. That is, $\nu_\epsilon \models_{\mathcal{L}} \Phi^k$ and $\nu_\epsilon \models_{\mathcal{L}} \Psi^k$ for infinitely many indices $k \in \mathbb{Z}$ with $\mathbb{Z} \subseteq I$.

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First, $\nu_\epsilon \models_{\mathcal{L}} \Phi^k$ implies that for $k \in \mathbb{Z}$ and $i \leq k$,

$$p_i \leq \nu_\epsilon([\![\Phi_i^k]\!]_{\mathcal{L}}) \leq \nu_\epsilon([s_i]) \vee \nu_\epsilon(\bigcup_{m \in I \wedge m > k} [s_m]).$$

Hence,

$$p_i \leq \inf_{k \in \mathbb{Z}} \nu_\epsilon([\![\Phi_i^k]\!]_{\mathcal{L}}) \leq \nu_\epsilon([s_i]) \vee \inf_{k \in \mathbb{Z}} \nu_\epsilon(\bigcup_{m \in I \wedge m > k} [s_m]).$$

It then follows from (2) that $p_i \leq \nu_\epsilon([s_i])$ for each $i \leq k$. Further, $\mathbb{Z}$ is infinite implies that for each $i \in I$,

$$\mu([s_i]) \leq \nu_\epsilon([s_i]). \tag{3}$$

Second, $\nu_\epsilon \models_{\mathcal{L}} \Psi^k$ implies that for $k \in \mathbb{Z}$ and $i \leq k$,

$$\nu_\epsilon([\![\Phi_i^l]\!]_{\mathcal{L}}) < p_i + \epsilon = \mu([s_i]) + \epsilon.$$

Again by the left part of the Equation (1) and $\mathbb{Z}$ being infinite, we have that for each $i \in I$,

$$\nu_\epsilon([s_i]) < \mu([s_i]) + \epsilon. \tag{4}$$

Combining (3) and (4), we have that, for each $i \in I$,

$$\mu([s_i]) \leq \nu_\epsilon([s_i]) < \mu([s_i]) + \epsilon. \tag{5}$$

In summary, we have shown that, for any given ϵ, a ν_ϵ exists such that the inequalities in (5) holds. Thanks to image-finiteness, the Pigeonhole principle applies again and a transition $t \xrightarrow{a} \nu$ exists such that, for infinitely many ϵ,

$$\mu([s_i]) \leq \nu([s_i]) < \mu([s_i]) + \epsilon. \tag{6}$$

Since ϵ can be arbitrarily small, we have that $\mu([s_i]) = \nu([s_i])$ for each $i \in I$, as desired. □

The benefit of logical characterization is to help us find a witness formula when two states are not bisimilar.

Example 5.3. Consider the Fig. 1. Let $\varphi = \langle a \rangle([\langle b \rangle[\top]_1]_{0.2} \wedge [\langle d \rangle[\top]_1]_{0.6})$. Then it can be verified that $s \models_{\mathcal{L}} \varphi$ but $t \not\models_{\mathcal{L}} \varphi$. Hence s and t are not bisimilar.

Remark 5.4. (1) The proof of Theorem 5.2 is inspired by that of Theorem 6.3 [18]. However, it is more difficult because we can directly obtain $\mu([s_i]) = \nu([s_i])$ from $\mu([s_i]) \leq \nu([s_i])$ (the left part in Equation (6)) in [18]. The reason is as follows. Suppose that there exists an $i_0 \in I$ such that $\mu([s_{i_0}]) < \nu([s_{i_0}])$, then we have that $\sum_{i \in I} \mu([s_i]) < \sum_{i \in I} \nu([s_i])$, which gives rise to a contradiction, as in this case, μ and ν are probability distributions and then $\sum_{i \in I} \mu([s_i]) = \sum_{i \in I} \nu([s_i]) = 1$.

(2) Now, we continue explaining why the logic $\mathcal{L}^{bi}$ cannot characterize bisimilarity for **NFTS**s. The reason is that the logic $\mathcal{L}^{bi}$ is not enough to deal with the formula $\langle a \rangle(\Phi^k \wedge \Psi^k)$ that plays a key role in proving Theorem 5.2. In order to obtain the expected result of Theorem 5.2, we need to decompose the formula $\langle a \rangle_p \varphi$ in the logic $\mathcal{L}^{bi}$ into two formulae, one is $\langle a \rangle \psi$ that is interpreted over states, the other is $[\varphi]_p$ that is interpreted over distributions. In addition, we can only depend on state formulae to characterize bisimilarity for **NFTS**s, if the formula $\langle a \rangle_p \varphi$ is changed into the formula $\langle a \rangle[(p_i, q_i), (\phi_i, \varphi_i)]_{i=1}^n$. The $s \models \langle a \rangle[(p_i, q_i), (\phi_i, \varphi_i)]_{i=1}^n$ means that a transition $s \xrightarrow{a} \mu$ exists such that $\mu([\![\phi_i]\!]) \geq p_i$ and $\mu([\![\varphi_i]\!]) < q_i$ for all $1 \leq i \leq n$. Under this interpretation, we can also obtain the Equation (6) and then finish the proof of Theorem 5.2. The similar idea can be found in [10].

5.2. Real-valued logic

In this subsection, we focus on a real-valued logic that is also suited to characterize bisimilarity.
This part needs the implication operators satisfying the following condition: for any $a, b \in [0, 1]$,

$$a \Rightarrow b = 1 \text{ iff } a \leq b. \tag{7}$$

The following four implication operators satisfy this condition.

- Łukasiewicz implication: $a \Rightarrow b = \min(1 - a + b, 1)$;
- Gödel implication: $a \Rightarrow b = 1$ if $a \leq b$, and b otherwise;
- Goguen implication: $a \Rightarrow b = 1$ if $a = 0$, and $\min(\frac{b}{a}, 1)$ otherwise;
- Fodor implication [16] (also called R_0 implication [30]): $a \Rightarrow b = 1$ if $a \leq b$, and $\max(1 - a, b)$ otherwise.

The real-valued logic has the following formulae:

$$\begin{aligned} \varphi &::= \top \mid \varphi_1 \wedge \varphi_2 \mid \varphi \to p \mid p \to \varphi \mid \varphi \ominus p \mid \langle a \rangle \psi \\ \psi &::= \psi_1 \wedge \psi_2 \mid \psi \to p \mid p \to \psi \mid \varphi^\dagger \end{aligned}$$

where $a \in A$ and $p \in [0, 1]$. We write $\varphi \leftrightarrow p$ for $(\varphi \to p) \wedge (p \to \varphi)$. The φ ranges over the set of all state formulae $\mathcal{L}^s$, and the ψ ranges over the set of all distribution formulae $\mathcal{L}^d$.

Let us fix an **NFTS** $(S, A, \to)$ and interpret the formulae above.

Definition 5.5. A state formula $\varphi \in \mathcal{L}^s$ evaluates in $s \in S$ as follows:

- $[\![\top]\!](s) = 1$,
- $[\![\varphi_1 \wedge \varphi_2]\!](s) = \min([\![\varphi_1]\!](s), [\![\varphi_2]\!](s))$,
- $[\![\varphi \to p]\!](s) = [\![\varphi]\!](s) \Rightarrow p$,
- $[\![p \to \varphi]\!](s) = p \Rightarrow [\![\varphi]\!](s)$,
- $[\![\varphi \ominus p]\!](s) = \max([\![\varphi]\!](s) - p, 0)$,
- $[\![\langle a \rangle \psi]\!](s) = \max_{s \xrightarrow{a} \mu} [\![\psi]\!](\mu)$,

and a distribution formula $\psi \in \mathcal{L}^d$ evaluates in $\mu \in \mathcal{F}(S)$ as follows:

- $[\![\psi_1 \wedge \psi_2]\!](\mu) = \min([\![\psi_1]\!](\mu), [\![\psi_2]\!](\mu))$,
- $[\![\psi \to p]\!](\mu) = [\![\psi]\!](\mu) \Rightarrow p$,
- $[\![p \to \psi]\!](\mu) = p \Rightarrow [\![\psi]\!](\mu)$,
- $[\![\varphi^\dagger]\!](\mu) = \sup_{s \in S} \mu(s) \wedge [\![\varphi]\!](s)$.

Here conjunction and implication are interpreted as usual. The formula $\varphi \ominus p$ returns the difference $[\![\varphi]\!](s) - p$ if $[\![\varphi]\!](s) \geq p$, and 0 otherwise for any $s \in S$. The formula $\langle a \rangle \psi$ specifies the property for a state to perform action a and result in a possible distribution to satisfy ψ. Because of nondeterminism, from s there may be several transitions labeled by the same action a, e.g. $s \xrightarrow{a} \mu_i$ with $i \in I$. We take the optimal case by letting $[\![\langle a \rangle \psi]\!](s)$ be the maximal $[\![\psi]\!](\mu_i)$ when i ranges over I. Each state formulae φ immediately induces a distribution formula $\varphi^\dagger$.

Main differences between our real-valued logic and those for probabilistic systems (see for example, [2,12]) are that our real-valued logic includes implication formulae and it can characterize bisimilarity under a class of fuzzy logics satisfying the Equation (7).

The following lemma is important for proving Theorem 5.7.

Lemma 5.6. *Let $\mathcal{M} = (S, A, \to)$ be finitary and $G = \{[s_i] \mid i \in I\}$ be the set consisting of all equivalence classes of the equivalence relation R, where $R = \{(s, t) \mid [\![\varphi]\!](s) = [\![\varphi]\!](t), \varphi \in \mathcal{L}^s\}$. If for any $\mu, \nu \in \mathcal{F}(S)$, a $k_0 \in I$ exists such that $\mu([s_{k_0}]) \neq \nu([s_{k_0}])$, then a state formula $\varphi \in \mathcal{L}^s$ exists such that $[\![\varphi^\dagger]\!](\mu) \neq [\![\varphi^\dagger]\!](\nu)$.*

Proof. Note that G is finite since the $\mathcal{M}$ being finitary implies S is finite. Let $\mu, \nu \in \mathcal{F}(S)$ and $\mu([s_{k_0}]) \neq \nu([s_{k_0}])$. We consider two cases.

Case 1. G includes only one equivalence class $[s_{k_0}]$, then $[s_{k_0}] = S$. In this case, the state formula $\top$ satisfies $[\![\top^\dagger]\!](\mu) \neq [\![\top^\dagger]\!](\nu)$, since $[\![\top^\dagger]\!](\mu) = \mu(S) = \mu([s_{k_0}])$ and $[\![\top^\dagger]\!](\nu) = \nu(S) = \nu([s_{k_0}])$.

Case 2. G includes at least two equivalence classes. We first prove the following claim: for all $j \in I$ and $j \neq k_0$, a state formula $\varphi' \in \mathcal{L}^s$ exists such that

$$[\![\varphi']\!](s_{k_0}) = 1 \text{ and } [\![\varphi']\!](s_j) < 1. \qquad (*)$$

First, it is easy to see that for each $j \neq k_0$, a state formula $\varphi_{k_0 j} \in \mathcal{L}^s$ exists such that $[\![\varphi_{k_0 j}]\!](s_{k_0}) \neq [\![\varphi_{k_0 j}]\!](s_j)$ (otherwise $s_j \in [s_{k_0}]$).

Define:

$$\varphi'_{k_0 j} = \varphi_{k_0 j} \leftrightarrow [\![\varphi_{k_0 j}]\!](s_{k_0}).$$

Then for all $j \neq k_0$, $[\![\varphi'_{k_0 j}]\!](s_{k_0}) = 1$ and $[\![\varphi'_{k_0 j}]\!](s_j) < 1$.

Second, let

$$\varphi' = \bigwedge_{j \neq k_0} \varphi'_{k_0 j}.$$

Then φ' satisfies for all $j \neq k_0$,

$$[\![\varphi']\!](s_{k_0}) = 1 \text{ and } [\![\varphi']\!](s_j) < 1.$$

This completes the proof of the claim $(*)$ above.

Now, we prove that there exists a state formula φ such that $[\![\varphi^\dagger]\!](\mu) \neq [\![\varphi^\dagger]\!](\nu)$.

Since $\mu([s_{k_0}]) \neq \nu([s_{k_0}])$, without loss of generality, suppose that $\mu([s_{k_0}]) > \nu([s_{k_0}])$.

Taking $p > 0$ such that $\nu([s_{k_0}]) < 1 - p < \mu([s_{k_0}])$ and let $\varphi = \varphi' \ominus p$. Then it follows from the claim $(*)$ that

$$[\![\varphi]\!](s_{k_0}) = 1 - p \text{ and } [\![\varphi]\!](s_j) = \max([\![\varphi']\!](s_j) - p, 0) < 1 - p$$

for all $j \neq k_0$.

Now, we have that

$$\begin{aligned} [\![\varphi^\dagger]\!](\mu) &= [[\![\varphi]\!](s_{k_0}) \wedge \mu([s_{k_0}])] \vee [\sup_{j \neq k_0} [\![\varphi]\!](s_j) \wedge \mu([s_j])] \\ &= (1-p) \vee [\sup_{j \neq k_0} [\![\varphi]\!](s_j) \wedge \mu([s_j])] \\ &= 1 - p \ \ (\text{since } [\![\varphi]\!](s_j) < 1 - p \text{ for all } j \neq k_0), \end{aligned}$$

and,

$$\begin{aligned} [\![\varphi^\dagger]\!](\nu) &= [[\![\varphi]\!](s_{k_0}) \wedge \nu([s_{k_0}])] \vee [\sup_{j \neq k_0} [\![\varphi]\!](s_j) \wedge \nu([s_j])] \\ &= \nu([s_{k_0}]) \vee [\sup_{j \neq k_0} [\![\varphi]\!](s_j) \wedge \nu([s_j])] \\ &< 1 - p. \end{aligned}$$

The last step holds because $\nu([s_{k_0}]) < 1 - p$, $[\![\varphi]\!](s_j) < 1 - p$ for all $j \in I$ and $j \neq k_0$ and I is a finite set.

That is, we find a state formula $\varphi \in \mathcal{L}^s$ such that $[\![\varphi^\dagger]\!](\mu) \neq [\![\varphi^\dagger]\!](\nu)$. □

Another version of the logical characterization theorem is given as follows.

Theorem 5.7. *Let $\mathcal{M} = (S, A, \rightarrow)$ be finitary. For any $s, t \in S$, $s \sim t$ iff $[\![\varphi]\!](s) = [\![\varphi]\!](t)$ for all $\varphi \in \mathcal{L}^s$.*

Proof.

$(\Longrightarrow)$ Let $s \sim t$. Then we prove $[\![\varphi]\!](s) = [\![\varphi]\!](t)$ for any $\varphi \in \mathcal{L}^s$ by structure induction on φ. We only consider $\varphi \equiv \langle a \rangle \psi$, the other cases are immediate. We first prove the following fact: for any $\mu, \nu \in \mathcal{F}(S)$,

$$\mu \sim \nu \text{ implies that } [\![\psi]\!](\mu) = [\![\psi]\!](\nu) \text{ for any } \psi \in \mathcal{L}^d \tag{$**$}$$

where $\mu \sim \nu$ means that $\mu([s]) = \nu([s])$ for all $\sim$ equivalence classes $[s]$.

– $\psi \equiv \psi_1 \wedge \psi_2$, $\psi_1 \rightarrow p$, $p \rightarrow \psi_1$, which are straightforward by inductions on ψ_1 and ψ_2.

– $\psi \equiv \varphi_1^\dagger$. In this case,

$$\begin{aligned} [\![\varphi_1^\dagger]\!](\mu) &= \sup_{s' \in S} \mu(s') \wedge [\![\varphi_1]\!](s') \\ &= \sup_{s' \in S} \mu([s']_\sim) \wedge [\![\varphi_1]\!](s') \\ &= \sup_{s' \in S} \nu([s']_\sim) \wedge [\![\varphi_1]\!](s') \\ &= [\![\varphi_1^\dagger]\!](\nu). \end{aligned}$$

The second step holds because by induction on φ_1, we have that $[\![\varphi_1]\!](s') = [\![\varphi_1]\!](t')$ for all $t' \in [s']_\sim$, and the third step holds since $\mu \sim \nu$ and then $\mu([s']_\sim) = \nu([s']_\sim)$ for all $s' \in S$. We have proven the fact $(**)$ above.

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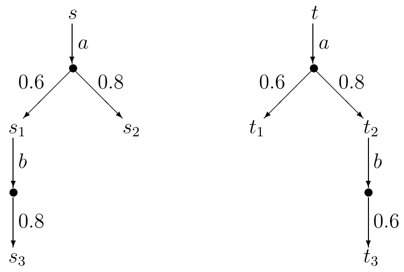

Fig. 2. s and t are distinguished by the real-valued logic.

Now, consider $\varphi \equiv \langle a\rangle\psi$. We have that $[\![\varphi]\!](s) = [\![\langle a\rangle\psi]\!](s) = \max_{s\xrightarrow{a}\mu}[\![\psi]\!](\mu)$. Since $s \sim t$, for any transition $s \xrightarrow{a} \mu$, a transition $t \xrightarrow{a} \nu$ exists such that $\mu(C) = \nu(C)$ for all $\sim$ equivalence classes C. That is, $\mu \sim \nu$. It follows from the fact $(**)$ that $[\![\psi]\!](\mu) = [\![\psi]\!](\nu)$. This leads to $[\![\varphi]\!](s) \le [\![\varphi]\!](t)$. On the other hand, the symmetry of $\sim$ implies that $[\![\varphi]\!](t) \le [\![\varphi]\!](s)$ also holds. Thus $[\![\varphi]\!](s) = [\![\varphi]\!](t)$ as desired.

($\Longleftarrow$) Let R and G be defined as Lemma 5.6. It suffices to prove that R is a bisimulation. Now, let $(s,t) \in R$ and $s \xrightarrow{a} \mu$. Then we need to prove that there exists a transition $t \xrightarrow{a} \nu$ such that $\mu([s_i]) = \nu([s_i])$ for all $i \in I$. Suppose for a contradiction that no a-transition from t satisfies this condition. Then for each ν_k $(k \in J)$ with $t \xrightarrow{a} \nu_k$, an equivalence class $[s_{i_k}] \in G$ exists such that $\mu([s_{i_k}]) \neq \nu_k([s_{i_k}])$. It follows from Lemma 5.6 that a state formula $\varphi_k \in \mathcal{L}^s$ exists such that $[\![\varphi_k^\dagger]\!](\mu) \neq [\![\varphi_k^\dagger]\!](\nu_k)$.

Define:

$$\psi_k = \varphi_k^\dagger \leftrightarrow [\![\varphi_k^\dagger]\!](\mu).$$

Then for all $k \in J$, $[\![\psi_k]\!](\mu) = 1$ and $[\![\psi_k]\!](\nu_k) < 1$.

Let $\varphi = \langle a\rangle \bigwedge_{k\in J} \psi_k$. Then we infer that

$$\begin{aligned}
[\![\varphi]\!](s) &= \max_{s\xrightarrow{a}\mu'}[\![\textstyle\bigwedge_{k\in J}\psi_k]\!](\mu') \\
&\ge [\![\textstyle\bigwedge_{k\in J}\psi_k]\!](\mu) \\
&= \min_{k\in J}[\![\psi_k]\!](\mu) \\
&= 1.
\end{aligned}$$

So $[\![\varphi]\!](s) = 1$. On the other hand,

$$\begin{aligned}
[\![\varphi]\!](t) &= \max_{t\xrightarrow{a}\nu_k}[\![\textstyle\bigwedge_{j\in J}\psi_j]\!](\nu_k) \\
&= \max_{t\xrightarrow{a}\nu_k}\min_{j\in J}[\![\psi_j]\!](\nu_k) \\
&\le \max_{t\xrightarrow{a}\nu_k}[\![\psi_k]\!](\nu_k) \\
&< 1 \ \ (\text{since } [\![\psi_k]\!](\nu_k) < 1 \text{ and } (S, A \to) \text{ is finitary}).
\end{aligned}$$

Hence $[\![\varphi]\!](s) \neq [\![\varphi]\!](t)$, which contradicts $(s,t) \in R$. We have finished the proof. □

Remark 5.8. (1) For **FTS**s, the two-sorted logic degenerates into the logic only including state formulae:

$$\varphi ::= \top \mid \varphi_1 \wedge \varphi_2 \mid \varphi \to p \mid p \to \varphi \mid \varphi \ominus p \mid \langle a\rangle\varphi.$$

(2) Note that conditions of Theorems 5.2 and 5.7 are different. The former requires that the transition system be image-finite but the state space can be countably infinite, while the latter requires that the transition system be finitary, which implies that the state space is finite.

Sometimes, it is convenient to determine whether two states are bisimilar by using this real-valued logic.

Example 5.9. Consider the Fig. 2. Let $\varphi = \langle a\rangle(\langle b\rangle\top^\dagger \ominus 0.6)^\dagger$. Then we have

$$[\![\varphi]\!](s) = 0.2 \text{ and } [\![\varphi]\!](t) = 0.$$

Hence s and t are not bisimilar.

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6. Logical metric

The behavioral distance given by Cao et al. [5] is a more robust way of formalizing similarity between fuzzy systems than bisimulations, which is defined as the greatest fixed point of some function. In this section, we present a new approach, i.e., a logical approach, to measuring similarity between fuzzy systems. With the help of the real-valued logic, we can easily reach this goal.

Definition 6.1. The logical distance between s and t is defined as follows:

$$d(s,t) = 1 - \inf_{\varphi \in \mathcal{L}^s} ([\![\varphi]\!](s) \Leftrightarrow [\![\varphi]\!](t)).$$

Theorem 5.7 tells us that under a finitary transition system, s and t are bisimilar iff $d(s,t) = 0$. Moreover, it is easy to see that $d(s,t)$ satisfies:

(D1) $d(s,s) = 0$;

(D2) $d(s,t) = d(t,s)$.

If we require further two conditions: the implication is a residual implication and has an adjoint t-norm $\otimes$ such that for any $a, b, c \in [0,1]$,

$$a \leq b \Rightarrow c \text{ iff } a \otimes b \leq c,$$

then, we can infer that

(D3) $d(s,t) \leq d(s,s') \oplus d(s',t)$ for any $s, s', t \in S$, where $\oplus$ is the dual t-conorm of the $\otimes$, defined as $a \oplus b = 1 - [(1-a) \otimes (1-b)]$ for any $a, b \in [0,1]$.

If the implication is taken as Łukasiewicz implication, then d is a pseudometric, and it is a pseudo-ultrametric [5] when the implication is the Gödel implication. In fact, d is a crisp relation under Gödel implication. i.e., $d(s,t) = 0$ or $d(s,t) = 1$ for any $s, t \in S$. More generally, we have the following proposition.

Proposition 6.2. *If the implication satisfies $a \Rightarrow 0 = 0$ for any $a > 0$, then d is a crisp relation.*

Proof. Suppose that $d(s,t) > 0$. Then a formula φ_1 exists such that $[\![\varphi_1]\!](s) \Leftrightarrow [\![\varphi_1]\!](t) < 1$. Define $\varphi_2 = \varphi_1 \leftrightarrow [\![\varphi_1]\!](s)$, we have that $[\![\varphi_2]\!](s) = 1$ and $[\![\varphi_2]\!](t) < 1$. Further, let $\varphi = \varphi_2 \ominus [\![\varphi_2]\!](t)$. Then we can get that $[\![\varphi]\!](s) > 0$ and $[\![\varphi]\!](t) = 0$. As a result, $[\![\varphi]\!](s) \Leftrightarrow [\![\varphi]\!](t) = 0$ that forces $d(s,t) = 1$. That is, d is a crisp relation. □

Hence, d is a crisp relation under Gödel and Goguen implications.

Let

$$\mathcal{S}(s,t) = 1 - d(s,t).$$

Then, we can get that: (1) $\mathcal{S}(s,s) = 1$; (2) $\mathcal{S}(s,t) = \mathcal{S}(t,s)$; and (3) $\mathcal{S}(s,s') \otimes \mathcal{S}(s',t) \leq \mathcal{S}(s,t)$. That is, $\mathcal{S}$ is a (generalized) similarity relation of Zadeh [35]. This says that the smaller the logical distance, the more states alike.

Example 6.3. Consider the Fig. 2 again, we can compute $d(s,t) = 0.2$ under Łukasiewicz and Fodor implications and hence $\mathcal{S}(s,t) = 0.8$, which shows that the similarity between processes s and t is 0.8.

Now let us consider the transitivity of bisimilarity. Since $\sim$ is an equivalence relation, we know that $s_1 \sim s_2$ and $s_2 \sim s_3$ imply $s_1 \sim s_3$. That is, $d(s_1,s_2) = 0$ and $d(s_2,s_3) = 0$ imply $d(s_1,s_3) = 0$. If we wish the logical distance to have a quantitative transitivity, i.e.,

$$d(s_1,s_2) \leq r \text{ and } d(s_2,s_3) \leq r \text{ imply } d(s_1,s_3) \leq r,$$

called r-transitivity, then the following proposition holds whose proof is straightforward and hence is omitted.

Proposition 6.4. *If the implication satisfies $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq a \Rightarrow c$ for any $a, b, c \in [0,1]$, then d has r-transitivity.*

It can be verified that d has r-transitivity under Fodor implication when $r < \frac{1}{2}$.

7. Conclusion and future work

We have given an $\mathcal{O}(|S|^4|{\rightarrow}|^2)$ algorithm to test bisimulations for **NFTS**s. We have already characterized bisimilarity for **NFTS**s by using two different modal logics soundly and completely. Moreover, we have defined a logical metric to capture similarity between states such that the smaller distance, the more states alike.

There are several problems that are worth further study. First, in the present article, the logical distance d is different from the behavioral distance d_f of Cao et al. [5] under the presented four implications. This is witnessed by Fig. 2, where $d_f(s,t) = 0.8$. It would be interesting to find an appropriate real-valued logic to characterize the behavioral distance d_f (see [12]). Second, apply our theory to model-checking of fuzzy systems (see for example, [23–25]). That is, given an **NFTS** $\mathcal{M}$ and a two-valued (resp. real-valued) state formula φ, determine whether $\mathcal{M}$ satisfies the two-valued formula φ (resp. what extent $\mathcal{M}$ satisfies the real-valued formula φ).

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