Linear Kernelizations for Restricted 3-Hitting Set Problems

Xuan Cai

[1] The Third Research Institute of Ministry of Public Security
Shanghai 201204, China
[2] BASICS Laboratory, Shanghai Jiao Tong University,
Shanghai 200240, China

In this paper, we study the Vertex Cover problem and its dual problem on bounded-degree, planar and quasi-regularizable 3-uniform hypergraphs. We provide linear kernelizations for these problems except for the Independent Set problem on quasi-regularizable 3-uniform hypergraphs. We show that this problem is W[1]-hard. Finally, we deduce lower bounds on the kernel size for the above problems from the parametric duality.

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a (simple) *hypergraph* with $\mathcal{V} = \{v_1, \cdots, v_n\}$ and $\mathcal{E} = \{B_1, \cdots, B_m\}$, where $v_i \neq v_j$ and $B_i \neq B_j$ for $i \neq j$. If $|B_i| = 1$ for some i, then we call it a *self-loop*. If $|B_i| = 2$ for any i, then $\mathcal{H}$ is an ordinary graph. In particular, if $\mathcal{H}$ permits k-*multiple* edges B_i with $k \geq 1$, namely, k copies of B_i, then we call $\mathcal{H}$ a *multi-hypergraph*.

Let v and w be two distinct vertices. If $w, v \in B_i$, then v is an *adjacent* vertex of w. $E(v) = \{B_i \in \mathcal{H} \mid v \in B_i\}$ is the *incidence set* of v in $\mathcal{H}$, and $|E(v)|$ is the *degree* of v, denoted by $\deg_{\mathcal{H}}(v)$. $\Delta_{\mathcal{H}}$ and $\delta_{\mathcal{H}}$ correspond to the *maximum* and the *minimum* degree of $\mathcal{H}$, respectively. If all the vertices of a hypergraph $\mathcal{H}$ have the same positive degree, i.e., $\Delta_{\mathcal{H}} = \delta_{\mathcal{H}} > 0$, then we call $\mathcal{H}$ *regular*.

Definition 1. *If the resulting hypergraph is regular after replacing each hyperedge B_i of $\mathcal{H}$ with k_i-multiple hyperedges ($k_i \geq 0$), then $\mathcal{H}$ is quasi-regularizable.*

A planar graph can be embedded on a plane without edge intersections except at the endpoints. Analogously, we can define a planar hypergraph.

Definition 2. *$G_{\mathcal{H}} = (V_1 \cup V_2, E)$ is a bipartite incidence graph of the hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ if $G_{\mathcal{H}}$ satisfies the following conditions:*

1. $V_1 = \mathcal{V}$.
2. $V_2 = \{v_B \mid B \in \mathcal{E}\}$.
3. $E = \{\{v, v_B\} \mid v \in V_1, v_B \in V_2, v \in B\}$.

Definition 3. *A hypergraph $\mathcal{H}$ is planar if $G_{\mathcal{H}}$ is planar.*

Definition 4. *A matching M in a graph $G = (V, E)$ is a subset of edges no two of which have a common endpoint. If no two edges of M are joined by an edge of G, then M is an induced matching.*

Theorem 1. *Let $\mathcal{H}$ be a 3-uniform hypergraph with bounded degree d. The* Vertex Cover *problem admits a problem kernel of size dk on $\mathcal{H}$.*

Theorem 2. *Let $\mathcal{H}$ be a 3-uniform hypergraph with bounded degree d. The* Independent Set *problem admits a problem kernel of size $(d\sqrt{2d+1}/3)k$ on $\mathcal{H}$.*

Theorem 3. *Let $\mathcal{H}$ be a planar 3-uniform hypergraph. The* Vertex Cover *problem admits a problem kernel of size $67k$ on $\mathcal{H}$.*

Theorem 4. *Let $\mathcal{H}$ be a planar 3-uniform hypergraph. The* Independent Set *problem admits a problem kernel of size $24k$ on $\mathcal{H}$.*

Theorem 5. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a planar 3-uniform hypergraph with $\mathcal{V} = \{v_1, \cdots, v_n\}$ and $\mathcal{E} = \{B_1, \cdots, B_m\}$. The* Induced Matching *problem admits a problem kernel of size $24k$ on $G_{\mathcal{H}}^{K}$.*

Theorem 6. *Let $\mathcal{H}$ be a quasi-regularizable 3-uniform hypergraph. The* Vertex Cover *problem admits a problem kernel of size $3k$ on $\mathcal{H}$.*

Theorem 7. *Let $\mathcal{H}$ be a quasi-regularizable 3-uniform hypergraph. The* Independent Set *problem is $W[1]$-hard on $\mathcal{H}$.*