Post's Simple Set

In his influential 1944 paper, Emil Post attempted to come up with an r.e. set whose Turing degree lies strictly between the recursive degree and the complete degree.

- E. Post. Recursively Enumerable Sets of Positive Integers and their Decision Problems, Bulletin of the American Mathematical Society 50:284-316, 1944.

Post's idea is to look for some 'simplicity' condition that violates the following property of a creative set:

- The complement of a creative set contains an infinite number of infinite r.e. set.

Simple Set

A set is immune if it is infinite but has no infinite r.e. subset.

A set is simple if it is r.e. and its complement is immune.

Fact. A simple set is neither recursive nor creative.

Post's Construction

Theorem (Post, 1944). There is a simple set.

Proof.
The idea is to construct by diagonalisation an immune set that is 'thin' enough not to contain any infinite r.e. set. □

Post's simple set: $rng(\phi_x(\mu z(\phi_x(z) > 2x)))$.

Hypersimple Set

An infinite set B is hyperimmune if for every disjoint strong array $\{F_n\}_{n\in\omega}$ there is some n such that $F_n \cap B = \emptyset$.

An r.e. set A is hypersimple if $\overline{A}$ is hyperimmune.

Hypersimple Set

Fact. A hypersimple set is simple.

A hyperimmune set A is 'thinner'. For every infinite r.e. set B the set $B \setminus A$ is infinite.

Weak Array

A uniformly recursively enumerable sequence $\{V_n\}_{n\in\omega}$ of r.e. sets is such that $\forall n \in \omega . V_n = W_{f(n)}$ for some recursive function f.

A sequence $\{F_n\}_{n\in\omega}$ of finite sets is a weak array if it is uniformly recursively enumerable.

Hyperhypersimple Set

An infinite set B is hyperhyperimmune if for every disjoint weak array $\{F_n\}_{n\in\omega}$ there is some n such that $F_n \cap B = \emptyset$.

An r.e. set is hyperhypersimple if $\overline{A}$ is hyperhyperimmune.

Hyperhypersimple Set

Fact. A hyperhypersimple set is hypersimple.

Simple sets, hypersimple sets and hyperhypersimple sets were introduced in Post's 1944 paper.

- Post knew that simple sets can be Turing complete.
- He suspected that hypersimple sets could be Turing complete.
- What he didn't know is that even hyperhypersimple sets may fail to be incomplete.

Majorizing Function

The principal function p_A of an infinite set $A = \{a_0 < a_1 < a_2 < \ldots\}$ is defined by $p_A(n) = a_n$.

A function f majorizes g if $f(x) \geq g(x)$ for all $x \in \omega$.

A function f majorizes an infinite set A if f majorizes p_A.

Theorem (Kuznecov, Medvedev, Uspenskii).

An infinite set is hyperimmune iff no recursive function majorizes it.

Proof.

($\Rightarrow$). Suppose f is a recursive function that majorizes p_A. Define

$$\begin{aligned} D_{g(0)} &= [0, f(0)], \\ D_{g(n+1)} &= [k, f(k)], \quad \text{where } k = 1 + \max\left(D_{g(n)}\right). \end{aligned}$$

Now $k \leq p_A(k) \leq f(k)$. Hence $p_A(k) \in D_{g(n+1)} \cap A$.

($\Leftarrow$). Suppose A is not hyperimmune. Then there is a disjoint strong array $\{D_{g(n)}\}_{n\in\omega}$ such that $D_{g(n)} \cap A \neq \emptyset$ for all $n \in \omega$.

Now set $f(x) = \max\left(\bigcup_{y\leq x} D_{g(y)}\right)$. Then f majorizes p_A. □

No total computable function grows faster than the principal function of a hyperimmune set. They are really ‘thin’.

Corollary. An infinite r.e. set is hypersimple iff no recursive function majorizes its complement.

Deficiency Set

Suppose A is a nonrecursive r.e. set enumerated by a 1-1 recursive function $f(x)$. Let

$$A_s = \{f(0), f(1), \ldots, f(s)\}.$$

We say that s is a deficient stage if $A_s \restriction f(s) \neq A \restriction f(s)$.

We say that s is a true stage if $A_s \restriction f(s) = A \restriction f(s)$.

Dekker Theorem

Theorem (Dekker, 1954). Every nonrecursive r.e. set is Turing equivalent to a hypersimple set.

Proof of Dekker Theorem

Suppose A is a nonrecursive r.e. set and $A = rng(f)$ for some 1-1 recursive function f. Let

$$a_s = f(s) \text{ and } A_s = \{f(0), f(1), \ldots, f(s)\}.$$

The deficiency set D of A for the enumeration f is defined by

$$D = \{s \mid \exists t. s < t \wedge a_t < a_s\}.$$

Clearly D is r.e., and both D and $\overline{D}$ are infinite.

Proof of Dekker Theorem

1. $A \leq_T D$. Notice that

$$x \in A \text{ iff } x \in \{a_0, a_1, \ldots, a_{p_{\overline{D}}(x)}\}$$

using the fact $x \leq a_{p_{\overline{D}}(x)}$.

2. $D \leq_T A$. To check $s \in D$ is to check $A_{s-1} = A \restriction a_s$.

3. If there was a recursive function that majorizes $p_{\overline{D}}$, then

$$x \in A \text{ iff } x \in \{a_0, a_1, \ldots, a_{g(x)}\},$$

which would imply that A is recursive. So $\overline{D}$ is hyperimmune. Conclude that D is hypersimple.

Freidberg (1957) and Yates (1965) introduced permitting method that allows to build up an r.e. set A recursive in a nonrecursive r.e. set B in such a way that x is introduced into A_s only if in a future stage $t \geq s$ a 'lesser' element y is introduced into B_t.

Permitting Method

Proposition. Suppose $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ are recursive enumerations of r.e. sets A and B such that $x \in A_{s+1} \setminus A_s$ implies $\exists y < x.y \in B \setminus B_s$. Then $A \leq_T B$.

Proof.

To decide if $x \in A$, find the least stage s such that $B_s{\restriction}x = B{\restriction}x$. Then $x \in A$ iff $x \in A_s$. □

Permitting Method

Theorem. Every nonrecursive r.e. set is Turing equivalent to a simple non-hypersimple set.

Proof.

Suppose B is a nonrecursive r.e. set and $B = rng(f)$ for some 1-1 recursive function f. Let

$$B_s = \{f(0), f(1), \ldots, f(s)\}.$$

We will construct a nonrecursive r.e. set A Turing equivalent to B.

- Let A_s denote the finite set constructed by the end of stage s. Suppose $\overline{A_s} = \{a_0^s < a_1^s < \ldots\}$ and $\overline{A} = \{a_0 < a_1 < \ldots\}$.

Continued on the next three slides. □

Permitting Method

Stage $s = 0$. Let $A_0 = \emptyset$.

Stage $s+1$. Construct A_{s+1} using permitting method.

1. This is to make sure that "A is simple" and "$A \leq_T B$".
   For each e ($\leq s$) such that $W_{e,s} \cap A_s = \emptyset$ and

$$\exists x.(x > 3e \wedge x \in W_{e,s} \wedge f(s+1) < x), \qquad (1)$$

   enumerate the least x satisfying (1).
   - Once x has been added to A_{s+1}, $W_{e,t} \cap A_t \neq \emptyset$ for all $t > s$, meaning that e will not be considered any more in Step 1.
2. Enumerate $a^s_{3f(s+1)+1}$. This is to make sure that "$B \leq_T A$".

Permitting Method

(a) A is not hypersimple.

Step 1 introduces at most s numbers in $A \cap [0, 3s]$.
Step 2 introduces at most s numbers in $A \cap [0, 3s]$.

So $\overline{A}$ is infinite and is majorized by the function $\lambda s.3s$.
Conclude that A is not hypersimple.

(b) A is simple.

Suppose e is minimal such that W_e is infinite and $W_e \cap A = \emptyset$.

Then the set $\{x \mid \exists s.(x \in W_{e,s} \wedge f(s+1) < x)\}$ would be finite, contradicting to the fact that B is nonrecursive.

- To check $y \in B$, compute some x, s such that $y \leq x \in W_{e,s}$ and $x \leq f(s+1)$, and then check $y \in \{f(0), \ldots, f(s+1)\}$.

Permitting Method

(c) $A \leq_T B$.

This is the proposition.

(d) $B \leq_T A$.

Given x, we can A-recursively compute $\mu s.a^s_{3x+1} = a_{3x+1}$.
Then $f(t) > x$ for all $t > s$. Hence $x \in B$ iff $x \in B_s$.

Effectively Simple Set

A simple set is effectively simple if there is a recursive function f such that

$$\forall e.(W_e \subseteq \overline{A} \Rightarrow |W_e| \leq f(e))$$

Fact. Post's simple set is effectively simple.

Effectively Simple Set are Complete

Theorem (Martin, 1966). An effectively simple set is complete.

Proof of Martin Theorem

Let $\{K_s\}_{s\in\omega}$ be a recursive enumeration of K, and

$$\theta(x) = \mu s.(x \in K_s).$$

Let A be effectively simple via f, $\{A_s\}_{s\in\omega}$ an enumeration of A.

$$\overline{A_s} = \{a_0^s < a_1^s < a_2^s < \ldots\} \text{ and } \overline{A} = \{a_0 < a_1 < a_2 < \ldots\}.$$

By the Recursion Theorem with parameters, one can define a recursive function $h(x)$ such that

$$W_{h(x)} = \begin{cases} \left\{a_0^{\theta(x)}, a_1^{\theta(x)}, \ldots, a_{fh(x)}^{\theta(x)}\right\}, & \text{if } x \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $r(x) = \mu s.(a^s_{fh(x)} = a_{fh(x)})$. Clearly $r \leq_T A$.

Proof of Martin Theorem

If $x \in K$ and $r(x) \leq \theta(x)$ then

$$W_{h(x)} \subseteq \overline{A} \text{ and } |W_{h(x)}| = fh(x) + 1,$$

contradicting to the assumption of f.

It follows that $r(x) > \theta(x)$ for all $x \in K$. So for all x

$$x \in K \text{ iff } x \in K_{r(x)}.$$

Consequently $K \leq_T A$.

Completeness Criterion for R.E. Set

Theorem (Arslanov, 1981). Suppose A is r.e.. Then A is complete iff there is a function $f \leq_T A$ such that $W_{f(x)} \neq W_x$ for all x.

Proof.

($\Rightarrow$). Recall that $\{x \mid W_x = \emptyset\}$ is complete. We may define f by

$$W_{f(x)} = \begin{cases} \emptyset, & \text{if } W_x \neq \emptyset, \\ \{0\}, & \text{if } W_x = \emptyset \end{cases}$$

using S-m-n Theorem. Clearly $W_{f(x)} \neq W_x$ for all x.

Continued on the next slide. □

Proof of Arslanov Theorem

($\Leftarrow$). Conversely, suppose $\forall x. W_{f(x)} \neq W_x$ and $f \leq_T A$.

By the Modulus Lemma there is a recursive function $\widehat{f}(x,s)$ such that $f(x) = \lim_s \widehat{f}(x,s)$ for every x, and $\{\lambda x.\widehat{f}(x,s)\}_{s\in\omega}$ has a modulus $m \leq_T A$.

Let $\{K_s\}_{s\in\omega}$ be an enumeration of K and $\theta(x) = \mu s.(x \in K_s)$.

By the Recursion Theorem with parameters there is an h such that

$$W_{h(x)} = \begin{cases} W_{\widehat{f}(h(x),\theta(x))}, & \text{if } x \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $x \in K$ and $\theta(x) \geq m(h(x))$, then $\widehat{f}(h(x),\theta(x)) = f(h(x))$ and $W_{f(h(x))} = W_{h(x)}$, contradicting to the assumption.

Hence $\forall x.(x \in K \Leftrightarrow x \in K_{m(h(x))})$. Conclude that $K \leq_T A$.

Remark on Arslanov Theorem

Corollary. Suppose $\mathbf{a}$ is an r.e. degree. Then $\mathbf{a} < \mathbf{0}'$ if and only if for every $f \in \mathbf{a}$ there exists n such that $W_n = W_{f(n)}$.

Arslanov Theorem is a generalization of Recursion Theorem.

- $W_n = W_{f(n)}$ is a generalization of $\phi_n = \phi_{f(n)}$.
- A total recursive function f satisfies $f < \mathbf{0}'$