The Parameterized Complexity of Maximality and Minimality Problems

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Abstract

Many parameterized problems (as the clique problem or the dominating set problem) ask, given an instance and a natural number k as parameter, whether there is a solution of size k. We analyze the relationship between the complexity of such a problem and the corresponding maximality (minimality) problem asking for a solution of size k maximal (minimal) with respect to set inclusion. As our results show many maximality problems increase the parameterized complexity, while "in terms of the W-hierarchy" minimality problems do not increase the complexity. We also address the corresponding construction, listing, and counting problems.

1. Introduction

Suppose we know (or at least have an upper bound on) the complexity of deciding whether a problem has a solution S of size k, that is, the solution S is a set of k elements. What can we say about the complexity of the existence of solutions of size k maximal (or minimal) with respect to set inclusion? Here a solution S is maximal with respect to set inclusion if there is no solution S' with $S \subset S'$. This paper studies such questions. By complexity we always mean the parameterized complexity, the parameter being the size of the solution.

Maximal and minimal solutions of combinatorial problems play an important role in various contexts, for example, in one of the earliest works in the area of worst-case analysis of NP-hard problems. In fact, Lawler's algorithm [25] for finding the chromatic number of a graph follows a simple dynamic programming approach, in which the chromatic number of each induced subgraph is computed by listing all its maximal independent sets. In a recent improvement due to Eppstein [11] the maximal independent sets of bounded size have to be listed. As a second example let us mention that TRANSVERSAL HYPERGRAPH (cf. [10]) is the problem of generating all satisfying assignments of minimal (Hamming) weight for a positive formula in CNF. For further examples and references see [3, 12, 20]. In parameterized complexity, various algorithms implement the listing of all minimal hitting sets of a given size, for example, the algorithm of the reduction yielding the Monotone Collapse Theorem mentioned below.

We start our analysis by mentioning three results, the first and the second one are well-known (cf. [13]) and the third one will be derived in Section 3 (see Proposition 11):

(i) The problem p-VERTEX-COVER ("Does a graph have a vertex cover of size k?") is fixed-parameter tractable and so is the problem p-MINIMAL-VERTEX-COVER ("Does a graph have a minimal vertex cover of size k?").

(ii) The problem p-INDEPENDENT-SET ("Does a graph have an independent set of size k?") is W[1]-complete and the problem p-MAXIMAL-INDEPENDENT-SET is W[2]-complete.

(iii) The problem p-DOMINATING-SET ("Does a graph have a dominating set of size k?") is W[2]-complete and so is p-MINIMAL-DOMINATING-SET.

So the minimality problems in (i) and (iii) do not increase the complexity while the maximality problem in (ii) does. As we show these are not isolated results but special cases of general phenomena: Many

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maximality problems increase the complexity, while "in terms of the W-hierarchy" minimality problems do not.

Let us first introduce a framework appropriate to discuss this type of questions. A set S of vertices of a graph $\mathcal{G}$ is a vertex cover if in $\mathcal{G}$ it satisfies the formula $\mathit{vc}(Z)$ of first-order logic with the set variable Z, where

$$\mathit{vc}(Z) := \forall x \forall y (\neg Exy \vee Zx \vee Zy)$$

(here the quantifiers range over the vertices, Exy means that there is an edge between x and y, and Zx means that x is an element of Z). We say that $\mathit{vc}(Z)$ Fagin-defines the problem p-VERTEX-COVER (on the class of graphs). Similarly the problems p-INDEPENDENT-SET and p-DOMINATING-SET are Fagin-defined by

$$\mathit{indep}(Z) := \forall x \forall y (\neg Exy \vee \neg Zx \vee \neg Zy) \quad \text{and} \quad \mathit{ds}(Z) := \forall y \exists x (Zx \wedge (x = y \vee Exy)),$$

respectively. Note that the formulas $\mathit{vc}(Z)$ and $\mathit{ds}(Z)$ are *positive* in Z (no occurrence of Z is in the scope of a negation symbol) and the formula $\mathit{indep}(Z)$ is *negative* in Z (every occurrence of Z is in the scope of exactly one negation symbol).

If $\varphi(Z)$ is an arbitrary first-order formula, we denote by p-WD$_\varphi$ the problem Fagin-defined by $\varphi(Z)$ (see Section 2.2 for the precise definition). It should be clear what we mean by p-MAXIMAL-WD$_\varphi$ and by p-MINIMAL-WD$_\varphi$. The problem p-MAXIMAL-DOMINATING-SET is trivial, since the set of all vertices is the only maximal dominating set in a given graph. Similarly, p-MINIMAL-INDEPENDENT-SET is trivial. More generally, one easily verifies (cf. Section 4) that the problem p-MAXIMAL-WD$_\varphi$ is trivial for $\varphi(Z)$ positive in Z and so is the problem p-MINIMAL-WD$_\varphi$ for $\varphi(Z)$ negative in Z.

We collect the main known results concerning the W-hierarchy and Fagin-definable problems (cf. [9, 18]). We use the following notation: If C is a class of parameterized problems, then $[\mathrm{C}]^{\mathrm{fpt}}$ denotes the class of problems (many-one) fpt-reducible to some problem in C.

Theorem 1. *Let $t \geq 1$.*

(a) $\mathrm{W}[t] = \big[\{p\text{-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula}\}\big]^{\mathrm{fpt}}$.

(b) If t is odd, then

$$\begin{aligned}\mathrm{W}[t] &= \big[\{p\text{-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula negative in } Z\}\big]^{\mathrm{fpt}} \\ &= \big[\{p\text{-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_{t+1}\text{-formula negative in } Z\}\big]^{\mathrm{fpt}}.\end{aligned}$$

(c) If t is even, then

$$\begin{aligned}\mathrm{W}[t] &= \big[\{p\text{-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula positive in } Z\}\big]^{\mathrm{fpt}} \\ &= \big[\{p\text{-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_{t+1}\text{-formula positive in } Z\}\big]^{\mathrm{fpt}}.\end{aligned}$$

The second equalities in (b) and (c) are formulations in terms of Fagin-definable problems of the Anti-monotone Collapse Theorem and the Monotone Collapse Theorem, respectively.

In this paper we first determine the complexity of some maximality and minimality problems which concern problems interesting in our context but not covered by our general results (Section 3). We then analyze maximality problems in Section sec:fdws. We observe that p-MAXIMAL-WD$_\varphi$ can be considerably harder than p-WD$_\varphi$. In fact, there is a Π_1-formula φ (hence $p\text{-WD}_\varphi \in \mathrm{W}[1]$) such that p-MAXIMAL-WD$_\varphi$ is W[P]-hard (see Theorem 19). In more conventional terms, we show that the maximal weighted satisfiability problem for formulas in 3-CNF is W[P]-hard (see Corollary 23).

We then turn to formulas $\varphi(Z)$ negative in Z (Section 6). For such formulas a solution of size k is already maximal, if no superset of it of size $k+1$ is a solution, too. Using this observation we derive the following theorem. A comparison with part (b) in Theorem 1 shows that the transition from p-WD$_\varphi$ to p-MAXIMAL-WD$_\varphi$ increases the complexity one level in the W-hierarchy; we already saw this phenomenon for the independent set problem in (ii).

Theorem 2. *If $t \geq 1$ is odd, then*

$$\begin{aligned} \mathrm{W}[t+1] &= \left[\{p\text{-}\textsc{Maximal}\text{-}\mathrm{WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula negative in } Z\}\right]^{\mathrm{fpt}} \\ &= \left[\{p\text{-}\textsc{Maximal}\text{-}\mathrm{WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_{t+1}\textit{-formula negative in } Z\}\right]^{\mathrm{fpt}}. \end{aligned}$$

This result implies, for example, that the maximal weighted satisfiability problem for formulas in 2-CNF with only negative literals is W$[2]$-complete (see Corollary 33).

We then consider minimality problems (Section 7). A comparison of (a) and (b) of the following theorem with (a) and (c) in Theorem 1, respectively, shows that for minimality problems we do not have an increase of complexity. Moreover, p-MINIMAL-WD$_\varphi$ is fixed-parameter tractable for every Π_1-formula $\varphi(Z)$.

Theorem 3. *(a) If $t \geq 2$, then*

$$\mathrm{W}[t] = \left[\{p\text{-}\textsc{Minimal}\text{-}\mathrm{WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula}\}\right]^{\mathrm{fpt}}.$$

(b) If $t \geq 2$ is even, then

$$\begin{aligned} \mathrm{W}[t] &= \left[\{p\text{-}\textsc{Minimal}\text{-}\mathrm{WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula positive in } Z\}\right]^{\mathrm{fpt}} \\ &= \left[\{p\text{-}\textsc{Minimal}\text{-}\mathrm{WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_{t+1}\textit{-formula positive in } Z\}\right]^{\mathrm{fpt}}. \end{aligned}$$

(c) p-MINIMAL-WD$_\varphi \in$ FPT *for every Π_1-formula $\varphi(Z)$.*

As we shall see in Example 46 there are Π_3-formulas $\varphi(Z)$ such that

$$p\text{-}\mathrm{WD}_\varphi \in \mathrm{FPT} \quad \text{and} \quad p\text{-}\textsc{Minimal}\text{-}\mathrm{WD}_\varphi \text{ is W[2]-complete.}$$

So what a comparison of Theorem 1 and Theorem 3 shows for minimality problems should be stated more precisely as: The quantifier complexity of $\varphi(Z)$ yields the same upper bounds for the complexity of p-MINIMAL-WD$_\varphi$ as for the complexity of p-WD$_\varphi$.

Let $d \geq 2$. We exemplify the consequences of our results on maximality and minimality problems for the weighted satisfiability problem of propositional formulas in $\Gamma_{t,d}$, $\Gamma^+_{t,d}$, and $\Gamma^-_{t,d}$ (these sets are defined in Section 2.3) in the following table:

	maximality problem	minimality problem
$\Gamma_{t,d}$	$t=1, d=2$: W$[2]$-complete otherwise: W[P]-hard	$t=1$: FPT $t>1$: W[t]-complete
$\Gamma^+_{t,d}$	FPT	t even: W$[t]$-complete t odd: W$[t-1]$-complete
$\Gamma^-_{t,d}$	t even: W$[t]$-complete t odd: W$[t+1]$-complete	FPT

We also address the corresponding construction problems (construct a maximal/minimal solution of $\varphi(Z)$ of size k) and listing problems (list all maximal/minimal solution of $\varphi(Z)$ of size k). What we obtain can be phrased as follows: If the corresponding decision problem is in W$[t]$, then the construction problem and the listing problem have an fpt (delay) algorithm with an oracle to a problem in W$[t]$.

We also consider problems "dual" to our maximality and minimality problems, namely the problems p-NON-MAXIMAL-WD$_\varphi$ and p-NON-MINIMAL-WD$_\varphi$ that ask for solutions of size k that are not maximal and not minimal, respectively. While non-minimal problems behave as the minimal problems (see Theorem 47), it turns out that non-maximal problems do not increase the complexity in the sense that p-NON-MAXIMAL-WD$_\varphi \in$ W$[t]$ for every Π_t-formula $\varphi(Z)$ negative in Z (see Theorem 38).

In view of the fact that

> the number of maximal solutions is the difference between the number of all solutions and the number of non-maximal solutions, (1)

we started to study whether our results generalize to the corresponding counting problems. If so, then for odd $t \geq 1$ Theorem 2 together with equation (1) would imply that some $\#\mathrm{W}[t+1]$-complete problem is solvable by an fpt-algorithm with an oracle to some problem in $\#\mathrm{W}[t]$. Unfortunately, this is not the case: While our results for the maximality and minimality generalize to the counting context (see Theorems 52 and 53, respectively), this is only partly true for non-minimality and non-maximality problems (see Theorems 54 and 64, respectively). We address the extensions for counting in Section 9.

We finish this introduction with some remarks concerning the proof methods. Based on [9], in [17, 18] the relationship between weighted satisfiability problems for fragments of propositional logic, model-checking problem for fragments of first-order logic, and Fagin-definable problems has been analyzed systematically and corresponding "translation procedures" were developed. Partly, our proofs built on these procedures. Maybe the technically most difficult proof is that of part (a) of Theorem 2 (compare Proposition 31). We should mention that our results, in particular Theorem 2 and Theorem 3, remain true if Z is replaced by a relation symbol of arbitrary arity.

Some of the results in this paper were announced in [4].

2. Preliminaries

The set of natural numbers (that is, nonnegative integers) is denoted by $\mathbb{N}$. For a natural number n let $[n] := \{1, \ldots, n\}$.

2.1. Parameterized complexity. We assume that the reader is familiar with the basic notions of parameterized complexity theory (cf. [8, 18]). We denote by FPT the class of all fixed-parameter tractable problems. For parameterized problems $\mathcal{P}$ and $\mathcal{P}'$ we write $\mathcal{P} \leq^{\text{fpt}} \mathcal{P}'$ if there is a (many-one) fpt-reduction from $\mathcal{P}$ to $\mathcal{P}'$. We write $\mathcal{P} \equiv^{\text{fpt}} \mathcal{P}'$ if $\mathcal{P} \leq^{\text{fpt}} \mathcal{P}'$ and $\mathcal{P}' \leq^{\text{fpt}} \mathcal{P}$, and we write $\mathcal{P} <^{\text{fpt}} \mathcal{P}'$ if $\mathcal{P} \leq^{\text{fpt}} \mathcal{P}'$ but not $\mathcal{P}' \leq^{\text{fpt}} \mathcal{P}$. By $[\mathcal{P}]^{\text{fpt}}$ we denote the class of problems fpt-reducible to $\mathcal{P}$. Similarly (and as already mentioned in the Introduction), if C is a class of parameterized problems, $[\mathrm{C}]^{\text{fpt}}$ is the class of problems fpt-reducible to some problem in C.

2.2. First-order logic. A *vocabulary* τ is a finite set of relation symbols. Each relation symbol has an *arity*. A τ*-structure* $\mathcal{A}$ consists of a set A called the *universe*, which we assume to be finite, and an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r-ary relation symbol $R \in \tau$. For example, we view a *graph* as a structure $\mathcal{G} = (G, E^{\mathcal{G}})$, where E is a binary relation symbol and $E^{\mathcal{G}}$ is an irreflexive and symmetric binary relation on the set of vertices G. Nevertheless, often we denote the vertex set of a graph $\mathcal{G}$ by V and the edge set by E (instead of G and $E^{\mathcal{G}}$) and use the set notation $\{v, w\}$ for edges.

Formulas of first-order logic of vocabulary τ are built up from atomic formulas $x = y$ and $Rx_1 \ldots x_r$ where $x, y, \bar{x}$ with $\bar{x} = x_1, \ldots, x_r$ are variables and $R \in \tau$ is of arity r using the boolean connectives $\neg$, $\wedge$, $\vee$ and existential and universal quantification. (The connectives $\rightarrow$ and $\leftrightarrow$ are understood as abbreviations.) *Literals* are atomic or negated atomic formulas. For $t \geq 1$, let Σ_t denote the class of all first-order formulas of the form

$$\exists x_{11} \ldots \exists x_{1k_1} \forall x_{21} \ldots \forall x_{2k_2} \ \ldots \ Qx_{t1} \ldots Qx_{tk_t} \ \psi,$$

where $Q = \forall$ if t is even and $Q = \exists$ otherwise, and where ψ is quantifier-free. Π_t-formulas are defined analogously starting with a block of universal quantifiers. Let $t, u \geq 1$. A formula φ is $\Sigma_{t,u}$, if it is Σ_t and all quantifier blocks after the leading existential block have length $\leq u$.

For a class Φ of first-order formulas we consider the *parameterized model-checking problem* (by $|\varphi|$ we denote the length of the formula φ):

p-MC(Φ)	
Input:	A structure $\mathcal{A}$ and a sentence $\varphi \in \Phi$.
Parameter:	$\lvert\varphi\rvert$.
Question:	Does $\mathcal{A} \models \varphi$ hold, that is, is $\mathcal{A}$ a model of φ?

Let Z be a fixed set variable (that is, unary relation variable). We consider first-order formulas that may contain atomic subformulas of the form Zx. For $t, d \geq 1$ we denote by $\Pi_{t/d}$ the set of Π_t-formulas $\varphi(Z)$, where for even (odd) t

$$\varphi = \forall \bar{x}_1 \exists \bar{x}_2 \ldots \exists \bar{x}_t \bigvee_{i \in I} \varphi_i \qquad \Big(\varphi = \forall \bar{x}_1 \exists \bar{x}_2 \ldots \forall \bar{x}_t \bigwedge_{i \in I} \varphi_i\Big),$$

and each φ_i is a conjunction (disjunction) of literals with at most d occurrences of Z. Of course, every Π_t-formula with at most d occurrences of Z is equivalent to a $\Pi_{t/d}$-formula.

A first-order formula $\varphi = \varphi(Z)$ *Fagin-defines* the problem:

p-WD$_\varphi$	
Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is there a subset $S \subseteq A$ of size k such that $\mathcal{A} \models \varphi(S)$?

2.3. Propositional logic. Formulas of propositional logic are built up from propositional variables X, Y, $X_1, X_2, \ldots$ by taking conjunctions, disjunctions, and negations. We distinguish between *small conjunctions*, denoted by $\wedge$, which are just conjunctions of two formulas, and *big conjunctions*, denoted by $\bigwedge$, which are conjunctions of nonempty finite sets of formulas. Analogously, we distinguish between *small disjunctions*, $\vee$, and *big disjunctions*, $\bigvee$. In the context of propositional logic *literals* are propositional variables or negated propositional variables.

For $t \geq 0$ and $d \geq 1$ we inductively define the following classes $\Gamma_{t,d}$ and $\Delta_{t,d}$ of formulas:

$$\begin{aligned}
\Gamma_{0,d} &:= \{\lambda_1 \wedge \ldots \wedge \lambda_s \mid s \in [d],\ \lambda_1, \ldots, \lambda_s \text{ literals}\},\\
\Delta_{0,d} &:= \{\lambda_1 \vee \ldots \vee \lambda_s \mid s \in [d],\ \lambda_1, \ldots, \lambda_s \text{ literals}\},\\
\Gamma_{t+1,d} &:= \Big\{ \bigwedge_{i \in I} \delta_i \mid I \text{ a finite nonempty index set and } \delta_i \in \Delta_{t,d} \text{ for all } i \in I \Big\},\\
\Delta_{t+1,d} &:= \Big\{ \bigvee_{i \in I} \gamma_i \mid I \text{ a finite nonempty index set and } \gamma_i \in \Gamma_{t,d} \text{ for all } i \in I \Big\}.
\end{aligned}$$

If in the definition of $\Gamma_{0,d}$ and $\Delta_{0,d}$ we require that all literals are positive (negative), then we obtain the sets denoted by $\Gamma^+_{t,d}$ and $\Delta^+_{t,d}$ ($\Gamma^-_{t,d}$ and $\Delta^-_{t,d}$), respectively.

We denote by $\text{Var}(\alpha)$ the set of propositional variables of a propositional formula α. Let V be a set of propositional variables. Often we tacitly identify an assignment $S : V \to \{\text{TRUE}, \text{FALSE}\}$ with the set $\{X \in V \mid S(X) = \text{TRUE}\}$. The *weight* of an assignment S is $|S|$, the number of variables set to TRUE. A propositional formula α is *k-satisfiable* (where $k \in \mathbb{N}$), if there is an assignment for the set of variables of α of weight k satisfying α. For a set Γ of propositional formulas, the *parameterized weighted satisfiability problem p-WSAT(Γ) for formulas in* Γ is the following problem:

p-WSAT(Γ)	
Input:	A propositional formula $\alpha \in \Gamma$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is α k-satisfiable?

For $t \geq 0$, $d \geq 1$, and $\alpha \in \Gamma^+_{t,d} \cup \Gamma^-_{t,d}$ we define the *parse structure* $\mathcal{A}_{\text{parse}}(\alpha)$ *of* α. This is a structure of vocabulary $\tau_{\text{parse}} := \{E, \mathit{FIR}, \mathit{VAR}\}$, where E is binary and *FIR* and *VAR* are unary. We obtain $\mathcal{A}_{\text{parse}}(\alpha)$ from the parse tree of α, where $E^{\mathcal{A}_{\text{parse}}(\alpha)}$ is the edge relation with edges directed from the root to the leaves, by the following manipulations:

– first we contract the edges between a negative literal and its variable and identify the node obtained thereby with the node of the variable (clearly this is only necessary for formulas in $\Gamma^-_{t,d}$);

– we identify the nodes corresponding to the same propositional variable and identify the node obtained thereby with the variable itself;

– we set $\mathit{FIR}^{\mathcal{A}_{\text{parse}}(\alpha)} := \{u \mid u \text{ is a child of the root of the parse tree of } \alpha\}$ and $\mathit{VAR}^{\mathcal{A}_{\text{parse}}(\alpha)} := \text{Var}(\alpha)$.

We also have to consider *circuits*. Here, circuits consist of *input nodes*, *and-nodes* and *or-nodes* of arity two, *negation-nodes* and they have exactly one *output node* (that is, we only consider Boolean circuits). We denote by CIRC the class of all circuits and by p-WSAT(CIRC) the weighted satisfiability problem for circuits. A circuit is *positive*, if it does not contain any negation-nodes. A circuit is *negative*, if every input node has out-degree 1 and is adjacent to a negation-node and there are no other negation-nodes.

2.4. The W-hierarchy and the class W[P]. The following theorem (e.g., see [18]) contains well-known characterizations (or definitions) of the W-hierarchy and the class W[P] in terms of model-checking problems and weighted satisfiability problems. Characterizations in terms of Fagin-definable problems were presented in Theorem 1.

Theorem 4. *(a) Let $t, u \geq 1$. Then p-MC$(\Sigma_{t,u})$ is W$[t]$-complete under fpt-reductions.*

(b) Let $t, d \geq 1$ and $t + d \geq 3$. Then p-WSAT$(\Gamma_{t,d})$ is W$[t]$-complete under fpt-reductions.

(c) Let $t, d \geq 1$ and $t + d \geq 3$. If t is even (odd), then p-WSAT$(\Gamma^{+}_{t,d})$ (p-WSAT$(\Gamma^{-}_{t,d})$) is W$[t]$-complete under fpt-reductions.

(d) p-WSAT(CIRC) is W[P]-complete under fpt-reductions.

3. Some examples

In this section we determine the complexity of some maximality and minimality problems which concern problems interesting in our context but not covered by our general results.

For a class Γ of propositional formulas or circuits, it should be clear what we mean by p-MINIMAL-WSAT(Γ) or by p-MAXIMAL-WSAT(Γ), as it was, say, for p-MINIMAL-DOMINATING-SET.

3.1. Maximality problems. Every $\Gamma_{1,1}$-formula α has at most one satisfying assignment, easily computable from α. Therefore:

Proposition 5. p-MAXIMAL-WSAT$(\Gamma_{1,1}) \in$ FPT.

We turn to $\Gamma_{1,2}$ and show:

Proposition 6. p-MAXIMAL-WSAT$(\Gamma_{1,2})$ *(and* p-MAXIMAL-WSAT$(\Gamma^{-}_{1,2})$*) are* W[2]*-complete under fpt-reductions.*

This result is due to Grohe [21]. We prove it by the following two lemmas. As mentioned in the Introduction, the problem p-MAXIMAL-INDEPENDENT-SET is W[2]-complete. [1] Therefore, the W[2]-hardness of p-MAXIMAL-WSAT$(\Gamma^{-}_{1,2})$ and hence of p-MAXIMAL-WSAT$(\Gamma_{1,2})$ is obtained by:

Lemma 7. p-MAXIMAL-INDEPENDENT-SET $\leq^{\text{fpt}}$ p-MAXIMAL-WSAT$(\Gamma^{-}_{1,2})$.

Proof: Let $(\mathcal{G}, k)$ with $\mathcal{G} = (V, E)$ be an instance of the first problem. We may assume that $\mathcal{G}$ has no isolated points. For $u \in V$ let X_u be a propositional variable. We set

$$\alpha := \bigwedge_{\{u,v\} \in E} (\neg X_u \vee \neg X_v).$$

For a subset $M \subseteq V$ we let $S(M) := \{X_u \mid u \in M\}$. One easily verifies that

$$M \text{ is an independent set of } \mathcal{G} \iff S(M) \text{ satisfies } \alpha.$$

[1] Note that p-MAXIMAL-INDEPENDENT-SET $=$ p-INDEPENDENT-DOMINATING-SET, where the latter problem, given a graph $\mathcal{G}$ and $k \in \mathbb{N}$ as parameter, asks for a dominating set of size k which additionally is an independent set. The W[2]-completeness of this result is already implicit in [7].

Thus, $(\mathcal{G}, k) \mapsto (\alpha, k)$ is the desired reduction, as α is a formula in $\Gamma^-_{1,2}$. □

By Theorem 4 (a) the following lemma shows that p-MAXIMAL-WSAT$(\Gamma_{1,2})$ (and hence p-MAXIMAL-WSAT$(\Gamma^-_{1,2})$) is in W[2].

Lemma 8. p-MAXIMAL-WSAT$(\Gamma_{1,2}) \leq^{\text{fpt}} p$-MC$(\Sigma_{2,2})$.

Proof: Let (α, k) be an instance of p-MAXIMAL-WSAT$(\Gamma_{1,2})$. We may assume that $k \geq 1$. We view unit clauses λ in α as $(\lambda_1 \vee \lambda_2)$ with $\lambda_1 = \lambda_2 = \lambda$.

Identify TRUE with 1 and FALSE with 0. For $i, j \in \{0, 1\}$ we define the binary relation $R_{i,j}$ on the variables of α by

$$R_{i,j} := \big\{(X, Y) \mid \text{by } \alpha, \text{"}X = i \text{ implies } Y = j \text{ in one step"}\big\}$$

more precisely,

$$R_{0,0} := \big\{(X, Y) \mid (X \vee \neg Y) \text{ is a conjunct of } \alpha\big\}, \quad R_{0,1} := \big\{(X, Y) \mid (X \vee Y) \text{ is a conjunct of } \alpha\big\},$$
$$R_{1,0} := \big\{(X, Y) \mid (\neg X \vee \neg Y) \text{ is a conjunct of } \alpha\big\}, \quad R_{1,1} := \big\{(X, Y) \mid (\neg X \vee Y) \text{ is a conjunct of } \alpha\big\}.$$

Now let the $T_{i,j}$ be the closure of the relations $R_{i,j}$ under the rules:

$$\begin{array}{rll}
R_{i,j}\, XY \to T_{i,j}\, XY & & \text{for } X, Y \in \text{Var}(\alpha) \text{ and } i, j \in \{0, 1\} \\
\to T_{0,0}\, XX & & \text{for } X \in \text{Var}(\alpha) \\
\to T_{1,1}\, XX & & \text{for } X \in \text{Var}(\alpha) \\
T_{i,j}\, XY \to T_{i',j'}\, YX & & \text{for } X, Y \in \text{Var}(\alpha) \text{ and } \{j, i'\} = \{j', i\} = \{0, 1\} \\
T_{i,j}\, XY,\ T_{j,\ell}\, YZ \to T_{i,\ell}\, XZ & & \text{for } X, Y, Z \in \text{Var}(\alpha) \text{ and } i, j, \ell \in \{0, 1\}.
\end{array}$$

Clearly, if an assignment S satisfies α, then

$$T_{i,j}\, XY \ \text{ and } \ S(X) = i \ \text{ imply } \ S(Y) = j. \tag{2}$$

Recall that Var(α) denotes the set of variables of α. Let $\mathcal{A} := \big(\text{Var}(\alpha), (R_{i,j})_{i,j\in\{0,1\}}, (T_{i,j})_{i,j\in\{0,1\}}\big)$. Note that $\mathcal{A}$ can be computed from α in polynomial time. For variables $x_1, \ldots, x_k, y$ we abbreviate by

$$y \in \{x_1, \ldots, x_k\} \quad \text{and} \quad y \notin \{x_1, \ldots, x_k\}$$

the quantifier-free formulas $\psi := \bigvee_{i\in[k]} y = x_i$ and $\neg\psi$, respectively. For variables $X_1, \ldots, X_k \in$ Var(α) we have

$$\{X_1, \ldots, X_k\} \text{ satisfies } \alpha \quad \iff \quad \mathcal{A} \models \chi_k(X_1, \ldots, X_k), \tag{3}$$

where

$$\begin{aligned}
\chi_k(x_1, \ldots, x_k) := \forall y \forall z \Big(& (R_{0,0}yz \to (y \notin \{x_1, \ldots, x_k\} \to z \notin \{x_1, \ldots, x_k\})) \\
& (R_{0,1}yz \to (y \notin \{x_1, \ldots, x_k\} \to z \in \{x_1, \ldots, x_k\})) \\
& (R_{1,0}yz \to (y \in \{x_1, \ldots, x_k\} \to z \notin \{x_1, \ldots, x_k\})) \\
& (R_{1,1}yz \to (y \in \{x_1, \ldots, x_k\} \to z \in \{x_1, \ldots, x_k\}))\Big).
\end{aligned}$$

We show that

$$(\alpha, k) \in p\text{-MAXIMAL-WSAT}(\Gamma_{1,2}) \iff \mathcal{A} \models \varphi_k,$$

where φ_k is

$$\exists x_1 \ldots \exists x_k \Big(\bigwedge_{i \neq j} \neg x_i = x_j \wedge \chi_k(x_1, \ldots, x_k) \wedge \forall y (T_{1,0}yy \vee \bigvee_{i\in[k]} y = x_i \vee \bigvee_{i\in[k]} T_{1,0}x_i y)\Big).$$

This yields the claimed reducibility, since φ_k is easily seen to be equivalent to a $\Sigma_{2,2}$-formula. The proof of the direction from right to left in the equivalence is easy using (2) and (3).

We turn to the other direction. Assume that $S = \{X_1, \ldots, X_k\}$ is a maximal satisfying assignment of α. We set

$$V := \Big\{ Y \mid T_{1,0}\ YY \vee \bigvee_{i\in[k]} T_{1,0}\ X_iY \Big\}.$$

We have to show that

$$S \cup V = \text{Var}(\alpha).$$

By contradiction, assume that $Y \in \text{Var}(\alpha)$ but Y is distinct from all X_i and is not contained in V. We define an assignment S' as follows:

$$S'(Z) := \begin{cases} \text{TRUE} & Z = X_i \text{ for some } i \in [k]\text{, or } T_{1,1}\ YZ, \\ \text{FALSE} & \text{otherwise.} \end{cases} \tag{4}$$

In particular, $S'(Y) = \text{TRUE}$ and therefore, $S \subset S'$. We show that S' satisfies α, which contradicts the maximality of S. Let $(\lambda_1 \vee \lambda_2)$ be a conjunct of α. We know that

$$S(\lambda_1) = \text{TRUE} \quad \text{or} \quad S(\lambda_2) = \text{TRUE}. \tag{5}$$

Now one analyzes the cases where λ_1, λ_2 contain a variable in S or a variable not in S separately, always deriving that $S'(\lambda_1 \vee \lambda_2) = \text{TRUE}$. This analysis is simple but tedious, we just present the argument for the (not completely trivial) case where

$$(\lambda_1 \vee \lambda_2) = (Z \vee \neg Z')$$

for variables Z, Z' with $Z \notin S$ and $Z' \notin S$. If $S'(Z') = \text{FALSE}$ then $S'(\lambda_1 \vee \lambda_2) = \text{TRUE}$ and we are done. Assume that $S'(Z') = \text{TRUE}$. Then, by (4), we have $T_{1,1}\ YZ'$. Furthermore, as $(Z \vee \neg Z')$ is a conjunct of α, we have $R_{0,0}\ ZZ'$. Hence, using the appropriate rules we obtain $T_{0,0}\ ZZ'$, $T_{1,1}\ Z'Z$, and finally $T_{1,1}\ YZ$. Therefore, by (4), we know that $S'(Z) = \text{TRUE}$; in particular, $S'(\lambda_1 \vee \lambda_2) = \text{TRUE}$. □

We close this subsection with an example, to which we will come back later.

Proposition 9. *p-MAXIMAL-CLIQUE is* W[2]*-complete under fpt-reductions.*

Proof: The usual reduction $((V,E),k) \mapsto \big((V, \{\{u,v\} \mid u \neq v,\ \{u,v\} \notin E\}), k\big)$ between p-CLIQUE and p-INDEPENDENT-SET is maximality preserving. The result now follows from point (ii) mentioned at the beginning of the Introduction. □

3.2. Minimality problems. While for $d \geq 2$ the problem p-WSAT$(\Gamma_{1,d})$ is W[1]-complete (cf. Theorem 4 (b)), we get:

Theorem 10. *p-MINIMAL-WSAT$(\Gamma_{1,d})$ is fixed-parameter tractable for every $d \geq 1$. Furthermore there is an algorithm that, given an instance (α, k) of p-WSAT$(\Gamma_{1,d})$ computes a list of all minimal satisfying assignments of size $\leq k$ in time $O(d^k \cdot k \cdot |\alpha|)$.*

Proof: Let $\alpha \in \Gamma_{1,d}$. Denote by $\mathcal{C}_\alpha$ the set of clauses of α. Then the algorithm LIST-MINIMALWSAT (on the next page) on input $(\mathcal{C}_\alpha, k)$ lists all minimal satisfying assignments of α of weight $\leq k$ in time $O(d^k \cdot k \cdot |\alpha|)$; in particular, p-MINIMAL-WSAT$(\Gamma_{1,d})$ is fixed-parameter tractable. The algorithm is similar to that enumerating all minimal hitting sets of size $\leq k$ in a hypergraph with hyperedges of size $\leq d$. Moreover, the proof of its correctness is similar. □

Finally we derive the result mentioned in (ii) at the beginning of the Introduction:

Proposition 11. *p-MINIMAL-DOMINATING-SET is* W[2]*-complete under fpt-reductions.*

LIST-MINIMAL-WSAT($\mathcal{C}, k$)
// $\mathcal{C}$ a set of clauses and $k \geq 0$.

if $\mathcal{C}$ contains the empty clause **then** return $\emptyset$
if $\mathcal{C}$ contains no clauses with only positive literals **then** return $\{\emptyset\}$
// i.e., the only minimal satisfying assignment is the all false assignment
if $k = 0$ **then** return $\emptyset$
// i.e., there is no satisfying assignment of weight 0,
// since $\mathcal{C}$ contains at least one clause with only positive literals
$c \leftarrow$ the “first” clause in $\mathcal{C}$ with only positive literals
$\mathcal{S} \leftarrow \emptyset$
for all $X \in c$ **do**
    $\mathcal{C}(X) \leftarrow \emptyset$
    // $\mathcal{C}(X)$ will collect all the (modified) clauses that are *not* already
    // satisfied when setting $X :=$ TRUE
    for all clauses $c' \in \mathcal{C} \setminus \{c\}$ **do**
        if $X \notin c'$ **then** $\mathcal{C}(X) \leftarrow \mathcal{C}(X) \cup \{c' \setminus \{\neg X\}\}$
    $\mathcal{S}(X) \leftarrow$ LIST-MINIMAL-WSAT($\mathcal{C}(X), k-1$)
    $\mathcal{S} \leftarrow \mathcal{S} \cup \big\{ s \cup \{X\} \mid s \in \mathcal{S}(X)$ and $s \cup \{X\}$ is a minimal
        satisfying assignment of $\mathcal{C} \big\}$.
return $\mathcal{S}$.

Proof: We first show that p-MINIMAL-DOMINATING-SET $\in$ W[2] by reducing it to p-MC($\Sigma_{2,1}$). For $k \in \mathbb{N}$ we have

$$(\mathcal{G}, k) \in p\text{-MINIMAL-DOMINATING-SET} \iff \mathcal{G} \models \varphi_k,$$

where (in φ_k the variables $x_1, \ldots, x_k$ correspond to the elements of a minimal dominating set and z_j witnesses that $\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k\}$ and hence all of its subsets are not dominating sets)

$$\varphi_k := \exists x_1 \ldots \exists x_k \exists z_1 \ldots \exists z_k \Big(\bigwedge_{\substack{i,j \in [k] \\ i<j}} x_i \neq x_j \wedge \forall y \bigvee_{i \in [k]} (x_i = y \vee E x_i y) \wedge \bigwedge_{j \in [k]} \bigwedge_{i \in [k], i \neq j} (x_i \neq z_j \wedge \neg E x_i z_j) \Big).$$

Since φ_k is (logical equivalent to) a $\Sigma_{2,1}$-sentence, this gives the desired reduction.

To show the W[2]-hardness of p-MINIMAL-DOMINATING-SET we present an fpt-reduction from the W[2]-complete problem p-DOMINATING-SET to it. Let $\mathcal{G} = (V, E)$ be a graph and $k \leq |V|$. We construct the graph $\mathcal{G}' = (V', E')$ as follows:

$$\begin{aligned} V' &:= \big(V \times [k]\big) \,\dot\cup\, V \,\dot\cup\, [k] \\ E' &:= \bigcup_{\ell \in [k]} \big\{ \{(u,\ell),(v,\ell)\} \mid u, v \in V, u \neq v \big\} \\ &\quad \cup \bigcup_{\ell \in [k]} \big\{ \{(u,\ell), v\} \mid u, v \in V \text{ and } (u = v \text{ or } \{u,v\} \in E) \big\} \\ &\quad \cup \big\{ \{(u,\ell), \ell\} \mid u \in V \text{ and } \ell \in [k] \big\}. \end{aligned}$$

One easily verifies that

$$\mathcal{G} \text{ has a dominating set of size } k \iff \mathcal{G}' \text{ has a minimal dominating set of size } k. \qquad \square$$

We close this section by mentioning that for the W[2]-complete problem p-KERNEL ("Does a directed graph $\mathcal{G}$ have a kernel of size k?") (cf [18]) we have

$$p\text{-KERNEL} = p\text{-MAXIMAL-KERNEL} = p\text{-MINIMAL-KERNEL},$$

since no proper subset or superset of a kernel is a kernel.

4. The framework

Contrary to p-WSAT(Γ) where it was clear what we mean by p-MINIMAL-WSAT(Γ) or by p-MAXIMAL-WSAT(Γ), for various problems it is not obvious what we mean by a maximal or minimal solution. For example, what is a maximal solution of the parameterized halting problem?

p-SHORT-NSTM-HALT	
Input:	A nondeterministic single-tape Turing machine $\mathbb{M}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Does $\mathbb{M}$ accept the empty string in k steps?

The situation is different for Fagin-defined problems.

Definition 12. Let $\varphi(Z)$ be a first-order formula of vocabulary τ. Let $\mathcal{A}$ be a τ-structure and $S \subseteq A$.

(a) S is a *solution (of $\varphi(Z)$ in $\mathcal{A}$)* if $\mathcal{A} \models \varphi(S)$.

(b) S is a *minimal solution (of $\varphi(Z)$ in $\mathcal{A}$)* if S is a solution and no subset $S' \subset S$ is a solution.

(c) S is a *maximal solution (of $\varphi(Z)$ in $\mathcal{A}$)* if S is a solution and no superset $S' \supset S$ is a solution.

We define the *maximality problem Fagin-defined by φ* (compare it with p-WD$_\varphi$ defined in Section 2.2):

p-MAXIMAL-WD$_\varphi$	
Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Does there exist a maximal solution of $\varphi(Z)$ in $\mathcal{A}$ of size k?

and the *minimality problem Fagin-defined by φ* as

p-MINIMAL-WD$_\varphi$	
Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Does there exist a minimal solution of $\varphi(Z)$ in $\mathcal{A}$ of size k?

In particular, on the class of graphs, the problem p-WD$_{indep}$ with

$$indep(Z) := \forall x \forall y(\neg Exy \vee \neg Zx \vee \neg Zy)$$

coincides with p-INDEPENDENT-SET and p-MAXIMAL-WD$_{indep}$ with p-MAXIMAL-INDEPENDENT-SET; moreover, the solutions of *indep*(Z) in a graph $\mathcal{G}$ are the independent sets.

Formulas $\varphi(Z)$ positive in Z are monotone, that is

$$\mathcal{A} \models \varphi(S) \text{ and } S \subseteq S' \subseteq A \text{ imply } \mathcal{A} \models \varphi(S').$$

Thus, if $\varphi(Z)$ is positive in Z, then there is at most one maximal solution S in a structure $\mathcal{A}$, namely $S = A$, and $\mathcal{A} \models \varphi(A)$ can be checked in polynomial time; therefore, p-MAXIMAL-WD$_\varphi$ is trivial for such φ.

Similarly, p-MINIMAL-WD$_\varphi$ is trivial for $\varphi(Z)$ negative in Z, since such formulas are antimonotone, that is

$$\mathcal{A} \models \varphi(S) \text{ and } S' \subseteq S \text{ imply } \mathcal{A} \models \varphi(S'),$$

and hence there is at most one minimal solution S in a structure $\mathcal{A}$, namely $S = \emptyset$.

As our exposition will show, the first equality of Theorem 2 even holds for Π_t-formulas antimonotone in Z. We stress that this remark only applies to Fagin-definable problems: There are antimonotone problems (in the sense that the subset of any solution is again a solution) that are in W$[1]$, but where it is not clear whether the corresponding maximality problem is in W$[2]$. Examples of such problems are p-IRREDUNDANT-SET and p-VC-DIMENSION. Both are in W$[1]$, and the corresponding maximality problems p-MAXIMAL-IRREDUNDANT-SET and p-MAXIMAL-SHATTERED-SET, respectively, are in W$^*[3] \cap$ A$[2]$ (cf. [2, 18]).

4.1. Variants. In some applications one is interested in the existence of a minimal solution of size $\leq k$ (and not of size exactly k) or as in the case of Eppstein's algorithm for faster exact coloring one has to list all minimal solutions of size $\leq k$. Let us denote by p-MINIMAL$^{\leq}$-WD$_\varphi$ the corresponding decision problem. Nearly all our results remain true if we replace p-MINIMAL-WD$_\varphi$ by p-MINIMAL$^{\leq}$-WD$_\varphi$. In fact, mostly our proofs can be adapted in a straightforward way and we leave that to the reader. We only point out in Remark 45 the changes that are necessary to get the analogue of our main result on minimality problems; more concretely, we show (the main step of a proof of):

$$\text{If } t \geq 2 \text{ and } \varphi(Z) \in \Pi_t, \text{ then } p\text{-MINIMAL}^{\leq}\text{-WD}_\varphi \in \text{W}[t]. \tag{6}$$

Nevertheless the problems p-MINIMAL-WD$_\varphi$ and p-MINIMAL$^{\leq}$-WD$_\varphi$ may have quite different complexities. Take for example

$$\varphi(Z) := \forall x \neg Zx \vee \forall y \exists x (Zx \wedge (y = x \vee Eyx)).$$

Then p-MINIMAL$^{\leq}$-WD$_\varphi \in$ FPT but p-MINIMAL-WD$_\varphi$ is W$[2]$-complete by Proposition 11.

Note also that for $\varphi(Z)$ positive in Z, we have for all structures $\mathcal{A}$ and all $k \leq |A|$ that

$$(\mathcal{A}, k) \in p\text{-MINIMAL}^{\leq}\text{-WD}_\varphi \iff (\mathcal{A}, k) \in p\text{-WD}_\varphi.$$

Hence, for such φ, the claim of (6) holds by Theorem 1 (a).

As already mentioned in the Introduction we also consider the problems "dual" to our maximality and minimality problems, namely the problems

p-NON-MAXIMAL-WD$_\varphi$	
Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is there a solution of $\varphi(Z)$ in $\mathcal{A}$ of size k which is *not* a maximal solution?

p-NON-MINIMAL-WD$_\varphi$	
Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is there a solution of $\varphi(Z)$ in $\mathcal{A}$ of size k which is *not* a minimal solution?

It should be clear what we mean by p-NON-MINIMAL-WSAT$(\Gamma_{t,d})$. Note that p-WSAT$(\Gamma_{t,d})$ is the union of p-MINIMAL-WSAT$(\Gamma_{t,d})$ and p-NON-MINIMAL-WSAT$(\Gamma_{t,d})$ in the sense that for every instance (α, k)

$$(\alpha, k) \in p\text{-WSAT}(\Gamma_{t,d}) \iff \begin{array}{l}(\alpha, k) \in p\text{-MINIMAL-WSAT}(\Gamma_{t,d}) \text{ or} \\ (\alpha, k) \in p\text{-NON-MINIMAL-WSAT}(\Gamma_{t,d})\end{array} \tag{7}$$

(of course, the same statement holds for "minimal" replaced by "maximal" and for problems p-WD$_\varphi$ instead of p-WSAT$(\Gamma_{t,d})$). In particular, p-WSAT$(\Gamma_{t,d})$ is fixed-parameter tractable if both, the corresponding minimality and the corresponding non-minimality problem, are in FPT. Thus, in view of Theorem 4 (b) and Theorem 10, we see that for all $d \geq 2$ the problem p-NON-MINIMAL-WSAT$(\Gamma_{1,d})$ cannot be in FPT unless FPT $=$ W$[1]$.

The union and intersection of problems in W$[t]$ are in W$[t]$, too. Since we did not find this result in the literature, we give a precise formulation and present a proof. In this context it is useful to view parameterized problems $\mathcal{P}$ as pairs (P, κ), where $P \subseteq \Sigma^*$ for a finite alphabet Σ and where $\kappa : \Sigma^* \to \mathbb{N}$ is a polynomial time computable function, the *parameterization* (cf. [18]).

Proposition 13. *Let $P, P' \subseteq \Sigma^*$ and let $\kappa : \Sigma^* \to \mathbb{N}$ be a parameterization. Then:*

(a) If $P \neq \emptyset$ and $(P', \kappa) \in$ FPT, then $(P \cup P', \kappa) \leq^{\text{fpt}} (P, \kappa)$.

(b) Let $t \geq 1$. If $(P, \kappa), (P', \kappa) \in$ W$[t]$, then $(P \cup P', \kappa) \in$ W$[t]$ and $(P \cap P', \kappa) \in$ W$[t]$.

Proof: (a) Let $x_+ \in P$ and let $\mathbb{A}$ be an fpt-algorithm solving (P', κ). The following procedure yields the desired fpt-reduction: Let $x \in \Sigma^*$. Check with $\mathbb{A}$ whether $x \in P'$. If so assign x_+ to x, otherwise assign x to itself.

(b) Let $(P, \kappa), (P', \kappa) \in$ W$[t]$. By Theorem 4 (a) there are fpt-reductions from (P, κ) and (P', κ) to p-MC$(\Sigma_{t,1})$, that is, there are fpt-algorithms associating with $x \in \Sigma^*$ structures $\mathcal{A}_x$ and $\mathcal{A}'_x$ and formulas φ_x and φ'_x in $\Sigma_{t,1}$ such that

$$(x \in P \iff \mathcal{A}_x \models \varphi_x) \quad \text{and} \quad (x \in P' \iff \mathcal{A}'_x \models \varphi'_x).$$

We may assume that $\mathcal{A}_x$ and $\mathcal{A}'_x$ are structures of the same vocabulary τ, that $A_x \cap A'_x = \emptyset$ and (adding dummy variables if necessary) that $\varphi_x = \exists x_1 \ldots \exists x_k \psi(\bar{x})$ and $\varphi'_x = \exists x_1 \ldots \exists x_k \psi'(\bar{x})$ with Π_{t-1}-formulas $\psi(\bar{x}), \psi'(\bar{x})$ and $\bar{x} = x_1 \ldots x_k$. Set $\sigma := \tau \cup \{U, U'\}$ with new unary relation symbols U and U'. Define the σ-structure $\mathcal{B}_x$ by

$$\begin{array}{rcll} B_x & := & A_x \cup A'_x & \\ R^{\mathcal{B}_x} & := & R^{\mathcal{A}_x} \cup R^{\mathcal{A}'_x} & \quad \text{(for } R \in \tau) \\ U^{\mathcal{B}_x} := A_x & \text{and} & U'^{\mathcal{B}_x} := A'_x. & \end{array}$$

Then

$$\begin{aligned} x \in (P \cup P') \quad &\iff \quad \mathcal{B}_x \models \exists x_1 \ldots \exists x_k \Big((\bigwedge_{i \in [k]} U x_i \wedge \psi^U(\bar{x})) \vee (\bigwedge_{i \in [k]} U' x_i \wedge \psi'^{U'}(\bar{x})) \Big) \\ x \in (P \cap P') \quad &\iff \quad \mathcal{B}_x \models \exists x_1 \ldots \exists x_k \exists y_1 \ldots \exists y_k \Big((\bigwedge_{i \in [k]} U x_i \wedge \psi^U(\bar{x})) \wedge (\bigwedge_{i \in [k]} U' y_i \wedge \psi'^{U'}(\bar{y})) \Big); \end{aligned}$$

here ψ^U for example, is obtained by relativizing the quantifiers to U, that is, by inductively replacing subformulas $\forall z \rho$ ($\exists z \rho$) by $\forall z (Uz \to \rho)$ ($\exists z (Uz \wedge \rho)$). This shows that $P \cup P'$ and $P \cap P'$ are both reducible to p-MC$(\Sigma_{t,2})$ and hence are in W$[t]$. □

Corollary 14. *For $d \geq 1$ we have p-WSAT$(\Gamma_{1,d}) \leq^{\text{fpt}} p$-NON-MINIMAL-WSAT$(\Gamma_{1,d})$.*

Proof: Since p-MINIMAL-WSAT$(\Gamma_{1,d}) \in$ FPT by Theorem 10, we get the claim from (7) and part (a) of the preceding proposition. □

5. Fagin-definable problems and weighted satisfiability problems

Some results in this paper will be easier to be derived for Fagin-definable problems and some for weighted satisfiability problems. The results can always be translated into the other framework by applying the well-known correspondence between weighted satisfiability problems and Fagin-definable problems stated in the following two lemmas, the first one translates Fagin-definable problems into weighted satisfiability problems and the second one contains a translation in the other direction. For proofs we refer the reader to [19, 14]. Recall the definition of $\Pi_{t/d}$-formulas of Subsection 2.2.

Lemma 15. *Let $t, d \geq 1$ and $\varphi(Z)$ be a $\Pi_{t/d}$-formula of vocabulary τ. Then there is a polynomial time algorithm associating with every τ-structure $\mathcal{A}$ a propositional formula $\alpha \in \Gamma_{t,d}$ such that $\mathrm{Var}(\alpha) \subseteq \{X_a \mid a \in A\}$ and for all $S \subseteq A$:*

$$\mathcal{A} \models \varphi(S) \iff \{X_b \mid b \in S\} \text{ satisfies } \alpha. \tag{8}$$

If $\varphi(Z)$ is positive (negative) in Z, then α can be chosen in $\Gamma^+_{t,d}$ (in $\Gamma^-_{t,d}$).

Lemma 16. *Let $t, d \geq 1$. There is a $\Pi_{t/d}$-formula $\varphi(Z)$ and a polynomial time algorithm associating with every propositional formula $\alpha \in \Gamma_{t,d}$ a structure $\mathcal{A}$ in a vocabulary τ containing a unary relation symbol VAR with $\mathit{VAR}^{\mathcal{A}} = \mathrm{Var}(\alpha)$ and such that for all $S \subseteq \mathrm{Var}(\alpha)$*

$$\mathcal{A} \models \varphi(S) \iff S \text{ satisfies } \alpha.$$

If we only consider formulas α in $\Gamma^+_{t,d}$ (α in $\Gamma^-_{t,d}$), then we can choose $\varphi(Z)$ as $\Pi_{t/d}$-formula positive (negative) in Z.

Corollary 17. *Let $t, d \geq 1$. Then:*

(a) $\left[\{p\text{-MAXIMAL-WD}_\varphi \mid \varphi(Z) \in \Pi_{t/d}\}\right]^{\text{fpt}} = \left[p\text{-MAXIMAL-WSAT}(\Gamma_{t,d})\right]^{\text{fpt}}$.

(b) $\left[\{p\text{-MINIMAL-WD}_\varphi \mid \varphi(Z) \in \Pi_{t/d}\}\right]^{\text{fpt}} = \left[p\text{-MINIMAL-WSAT}(\Gamma_{t,d})\right]^{\text{fpt}}$.

(c) $\left[\{p\text{-MAXIMAL-WD}_\varphi \mid \varphi(Z) \in \Pi_{t/d} \text{ negative in } Z\}\right]^{\text{fpt}} = \left[p\text{-MAXIMAL-WSAT}(\Gamma^-_{t,d})\right]^{\text{fpt}}$.

(d) $\left[\{p\text{-MINIMAL-WD}_\varphi \mid \varphi(Z) \in \Pi_{t/d} \text{ positive in } Z\}\right]^{\text{fpt}} = \left[p\text{-MINIMAL-WSAT}(\Gamma^+_{t,d})\right]^{\text{fpt}}$.

Proof: All proofs are similar, so we present that for (c). First let $\varphi(Z) \in \Pi_{t/d}$ be negative in Z and $(\mathcal{A}, k)$ an instance of p-MAXIMAL-WD$_\varphi$. Choose $\alpha \in \Gamma^-_{t,d}$ according to Lemma 15. Let S_0 be the set of all $a \in A$ such that the variable X_a does not occur in α and let $\ell_0 := |S_0|$. Then, by (8), for all $S \subseteq A$

$$\mathcal{A} \models \varphi(S) \quad \iff \quad \mathcal{A} \models \varphi(S \cup S_0). \tag{9}$$

The equivalences (8) and (9) show that

$$(\mathcal{A}, k) \in p\text{-MAXIMAL-WD}_\varphi \iff (\alpha, k - \ell_0) \in p\text{-MAXIMAL-WSAT}(\Gamma^-_{t,d}),$$

which yields a reduction from p-MAXIMAL-WD$_\varphi$ to p-MAXIMAL-WSAT$(\Gamma^-_{t,d})$.

Conversely, one gets a reduction from p-MAXIMAL-WSAT$(\Gamma^-_{t,d})$ to p-MAXIMAL-WD$_\psi$ for some $\Pi_{t/d}$-formula $\psi(Z)$ negative in Z using Lemma 16; in fact, we can choose as $\psi(Z)$ the formula $\varphi(Z) \wedge \forall x(Zx \rightarrow \mathit{VAR}\,x)$, where the formula $\varphi(Z)$ negative in Z is chosen according to Lemma 16. □

We state a first application of the translation procedures:

Proposition 18. *(a) If $\varphi(Z) \in \Pi_{1/1}$, then p-MAXIMAL-WD$_\varphi \in$ FPT.*

(b) $\mathrm{W}[2] = \left[\{p\text{-MAXIMAL-WD}_\varphi \mid \varphi(Z) \in \Pi_{1/2}\}\right]^{\text{fpt}}$.

(c) *If $\varphi(Z) \in \Pi_{1/d}$ for some $d \geq 1$, then there is an algorithm that on input $(\mathcal{A}, k)$ lists all minimal solutions of $\varphi(Z)$ in $\mathcal{A}$ of size k in fpt-time.*

Proof: Part (a) is a translation of Proposition 5, part (b) of Proposition 6, and part (c) of Theorem 10. □

Contrary to parts (a) and (b) of the preceding proposition, our next result shows that p-MAXIMAL-WD$_\varphi$ can have very high complexity for a $\Pi_{1/3}$-formula $\varphi(Z)$.

Theorem 19. *There exists a formula $\varphi(Z) \in \Pi_{1/3}$ such that p-MAXIMAL-WD$_\varphi$ is* W[P]*-hard under fpt-reductions.*

Before we start with the proof of this result we state three simple facts we are going to use in it. We call an assignment not satisfying a circuit $\mathcal{C}$ an *unsatisfying* assignment of $\mathcal{C}$.

Lemma 20. *p-WUNSAT(CIRC) is* W[P]*-hard under fpt-reductions, where*

p-WUNSAT(CIRC)	
Input:	A circuit $\mathcal{C}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Does $\mathcal{C}$ have an *unsatisfying* assignment of weight k?

Proof: We know that p-WSAT(CIRC) is W[P]-complete. Therefore the claim follows from the fact that we can associate in polynomial time with every circuit $\mathcal{C}$ a circuit $\mathcal{C}'$ with the same input nodes such that for every assignment for $\mathcal{C}$ (and hence for $\mathcal{C}'$):

$$S \text{ satisfies } \mathcal{C} \iff S \text{ does not satisfy } \mathcal{C}'.$$ □

The proofs of the following two lemmas are straightforward and we omit them. Recall that a circuit is positive if it does not contain any negation-nodes.

Lemma 21. *There is a polynomial time algorithm associating with every circuit $\mathcal{C}$ with set $\mathcal{X}$ of input nodes a positive circuit $\mathcal{C}^+$ with set $\mathcal{X} \cup \{X^- \mid X \in \mathcal{X}\}$ of input nodes (the new input node X^- takes over the role of "$\neg X$") such that for every assignment S of $\mathcal{C}$ and the assignment S^+ of $\mathcal{C}^+$ given by*

$$S^+ := S \cup \{X^- \mid X \notin S\}. \tag{10}$$

we have

$$S \textit{ satisfies } \mathcal{C} \iff S^+ \textit{ satisfies } \mathcal{C}^+. \tag{11}$$

We view positive circuits as $\tau_0 := \{E, \mathit{AND}, \mathit{OR}, \mathit{OUT}\}$-structures, where E is binary (the edge relation of the directed graph underlying the circuit with edges directed from the output node to the input nodes and *AND*, *OR*, and *OUT* are unary (the set of and-nodes, of or-nodes and the set containing the output node).

Lemma 22. *Let*

$$\begin{aligned} \mathit{posisat}(Z) := \forall x \forall y \forall z \Big(& ((\mathit{AND}\, x \wedge Exy \wedge Zx) \to Zy) \\ & \wedge ((\mathit{OR}\, x \wedge Exy \wedge Exz \wedge y \neq z \wedge Zx) \to (Zy \vee Zz)) \Big). \end{aligned}$$

Then for every positive circuit $\mathcal{C}$ and all subsets S of the set $\mathcal{X}$ of input nodes of $\mathcal{C}$ we have:

$$\mathcal{C} \models \mathit{posisat}(S) \tag{12}$$

and

$$S \textit{ satisfies } \mathcal{C} \iff \textit{for some } T \textit{ containing the output node with } S = T \cap \mathcal{X}: \ \mathit{posisat}(T) \tag{13}$$

Proof of Theorem 19: We present a reduction from p-WUNSAT(CIRC) to p-MAXIMAL-WD$_\varphi$ for some formula $\varphi(Z) \in \Pi_{1/3}$.

Let $(\mathcal{C}, k)$ be an instance of p-WUNSAT(CIRC), and let $X_1, \dots X_n$ be the input nodes of $\mathcal{C}$. We turn $\mathcal{C}$ into a new circuit $\mathcal{C}'$ with set $\mathcal{X}'$ of input nodes, where

$$\mathcal{X}' := \big\{X_{i,j} \mid i \in [k] \text{ and } j \in [n]\big\},$$

by replacing in $\mathcal{C}$ each X_j by a circuit equivalent to $\bigvee_{i\in[k]} X_{i,j}$. Obviously for $j_1, \dots, j_k \in [n]$,

$$\{X_{j_1}, \dots, X_{j_k}\} \text{ satisfies } \mathcal{C} \iff \big\{X_{1,j_1}, \dots, X_{k,j_k}\big\} \text{ satisfies } \mathcal{C}'. \tag{14}$$

Now we pass to the positive circuit $(\mathcal{C}')^+$ according to Lemma 21. In particular, $(\mathcal{C}')^+$ has as set of input variables the set

$$\big\{X_{i,j} \mid i \in [k] \text{ and } j \in [n]\big\} \cup \big\{X^-_{i,j} \mid i \in [k] \text{ and } j \in [n]\big\}.$$

We may assume that:

$$\text{if } (\mathcal{C}')^+ \models \textit{posisat}(T) \text{ and } T \text{ contains the output node, then } |T| \geq k+1.^2 \tag{15}$$

We view the circuit $(\mathcal{C}')^+$ thus obtained as a $\tau^+ := \tau_0 \cup \{\textit{OLDIN}, \textit{DIFF}, \textit{NEG}\}$-structure, where τ_0 was introduced before Lemma 22 and the new symbols are interpreted by:

$$\begin{aligned}
\textit{OLDIN}^{(\mathcal{C}')^+} &:= \mathcal{X}' \ \ (= \big\{X_{i,j} \mid i \in [k] \text{ and } j \in [n]\big\})\\
\textit{DIFF}^{(\mathcal{C}')^+} &:= \big\{(X_{i,j}, X_{i',j'}) \mid i \neq i' \text{ and } j \neq j'\big\}\\
\textit{NEG}^{(\mathcal{C}')^+} &:= \big\{(X, X^-) \mid X \in \mathcal{X}'\big\}.
\end{aligned}$$

For any set T of nodes of $(\mathcal{C}')^+$ we have

$$\begin{aligned}
(\mathcal{C}')^+ \models \textit{outorin}(T) &\iff T \text{ contains the output node or } T \subseteq \mathcal{X}' && (16)\\
(\mathcal{C}')^+ \models \textit{cons}(T) &\iff T \text{ does not contain the output node} && (17)\\
&\qquad \text{or for all } X \in \mathcal{X}'\text{: } (X \in T \iff X^- \notin T)\\
(\mathcal{C}')^+ \models \textit{assign}(T) &\iff \text{for all distinct } X_{i,j}, X_{i',j'} \in T\text{, we have: } i \neq i' \text{ and } j \neq j', && (18)
\end{aligned}$$

where

$$\begin{aligned}
\textit{outorin}(Z) &:= \forall x \forall y \big((\textit{OUT}\,x \wedge \neg Zx \wedge Zy) \to \textit{OLDIN}\,y\big)\\
\textit{cons}(Z) &:= \forall x \forall y \forall z \Big(((\textit{OUT}\,x \wedge Zx \wedge \textit{NEG}\,yz \wedge Zy) \to \neg Zz)\\
&\qquad\qquad \wedge\ ((\textit{OUT}\,x \wedge Zx \wedge \textit{NEG}\,yz \wedge \neg Zy) \to Zz)\Big)\\
\textit{assign}(Z) &:= \forall x \forall y \big((Zx \wedge Zy \wedge \textit{OLDIN}\,x \wedge \textit{OLDIN}\,y \wedge x \neq y) \to \textit{DIFF}\,xy\big).
\end{aligned}$$

Let

$$\varphi(Z) := \big(\textit{posisat}(Z) \wedge \textit{outorin}(Z) \wedge \textit{cons}(Z) \wedge \textit{assign}(Z)\big).$$

Then, up to equivalence, $\varphi(Z) \in \Pi_{1/3}$. We claim the following equivalence (which yields the desired reduction):

$$(\mathcal{C}, k) \in p\text{-WUNSAT(CIRC)} \iff ((\mathcal{C}')^+, k) \in p\text{-MAXIMAL-WD}_\varphi.$$

Assume first that $\mathcal{C}$ has an unsatisfying assignment $S = \big\{X_{j_1}, \dots X_{j_k}\big\}$ of weight k. Let

$$S' := \big\{X_{1,j_1}, \dots X_{k,j_k}\big\}.$$

[2] For a circuit $\mathcal{D}$ let $\mathcal{D}^*$ be the circuit obtained by taking two copies of $\mathcal{D}$, which share all nodes but the output node, and by adding a new output node, which is an and-gate and receives its values from the output nodes of the copies. Repeating this process one gets a circuit as claimed.

Then, using (12), (16), (17), and (18), one easily verifies that $(\mathcal{C}')^+ \models \varphi(S')$. We show that S' is a maximal solution of $\varphi(Z)$ in $(\mathcal{C}')^+$. Let $T \supset S'$ and assume that

$$(\mathcal{C}')^+ \models \varphi(T).$$

In particular $(\mathcal{C}')^+ \models \mathit{assign}\big(T\big)$, hence

$$S' = T \cap \mathcal{X}'.$$

If T does not contain the output node, then, by (16), we have $T \subseteq \mathcal{X}'$, and thus $S' = T$, which contradicts $S' \subset T$. Hence T contains the output node. Then, by (17), $(S')^+ := S' \cup \{X^- \mid X \notin S'\}$ is the set of input nodes of $(\mathcal{C}')^+$ in T. By $(\mathcal{C}')^+ \models \mathit{posisat}\big(T\big)$, we get from (13) that $(S')^+$ satisfies $(\mathcal{C}')^+$ and hence by (11), S' satisfies $\mathcal{C}'$, which finally, by (14), implies that S satisfies $\mathcal{C}$, a contradiction.

Now assume that $((\mathcal{C}')^+, k) \in p$-MAXIMAL-WD$_\varphi$ and that T is a maximal solution of size k. If T contains the output node, then by (15) we have $|T| \geq k+1$, a contradiction. Thus T does not contain the output node. It then follows by (16) that $T \subseteq \mathcal{X}'$. Together with (18) we conclude that $T = \big\{X_{1,j_1}, \dots X_{k,j_k}\big\}$ for some pairwise distinct $j_1, \dots, j_k$. Now let

$$S := \big\{X_{j_1}, \dots, X_{j_k}\big\},$$

which is an assignment of $\mathcal{C}$ of weight k. It only remains to show that S does not satisfy $\mathcal{C}$. We assume otherwise. Then by (14), T is a satisfying assignment for $\mathcal{C}'$ and hence by (11), $T^+ := T \cup \{X^- \mid X \notin T\}$ is an assignment satisfying $(\mathcal{C}')^+$. Moreover by (13), there is some set U of nodes of $(\mathcal{C}')^+$ containing the output node and containing precisely the input nodes in T^+ such that

$$(\mathcal{C}')^+ \models \mathit{posisat}\big(U\big).$$

Clearly, $(\mathcal{C}')^+ \models (\mathit{outorin} \wedge \mathit{cons} \wedge \mathit{assign})(U)$. Hence $(\mathcal{C}')^+ \models \varphi(U)$, which contradicts the maximality of T. □

By Corollary 17 (a), the preceding theorem yields:

Corollary 23. *p-MAXIMAL-WSAT$(\Gamma_{1,3})$ is* W[P]*-hard.*

Proposition 24. *There exists a formula $\varphi(Z) \in \Pi_{1/3}$ such that for every circuit $\mathcal{C}$ we have*

$$\mathcal{C} \text{ is not satisfiable} \quad \Longleftrightarrow \quad (\mathcal{C}^+, 0) \in p\text{-MAXIMAL-WD}_\varphi \tag{19}$$

(here $\mathcal{C}^+$, the circuit assigned to $\mathcal{C}$ according to Lemma 21, is viewed as a structure in an appropriate vocabulary; see the proof for details). In particular, p-MAXIMAL-WD$_\varphi$ is hard for para-(co-NP) *under fpt-reductions.*

In [15] for every classical complexity class C, a class para-C of parameterized problems was defined as follows:

A parameterized problem $\mathcal{P} = (P, \kappa)$ with $P \subseteq \Sigma^*$ is in para-C if there exist an alphabet Π, a computable function $g : \mathbb{N} \to \Pi^*$, and a problem $X \subseteq \Sigma^* \times \Pi^*$ such that $X \in$ C and for all $x \in \Sigma^*$ we have

$$x \in P \iff \big(x, g(\kappa(x))\big) \in X.$$

It is not hard to show that para-PTIME = FPT.

Proof of Proposition 24: Let $\mathcal{C}$ be a circuit with set $\mathcal{X}$ of input nodes. According to Lemma 21 the positive circuit $\mathcal{C}^+$ has $\mathcal{X} \cup \{X^- \mid X \in \mathcal{X}\}$ as set of input nodes. We consider $\mathcal{C}^+$ as $\tau_1 := \{E, \mathit{AND}, \mathit{OR}, \mathit{OUT}, \mathit{NEG}\}$-structure, where

$$\mathit{NEG}^{\mathcal{C}^+} := \big\{(X, X^-) \mid X \in \mathcal{X}\big\}$$

is defined as in the previous proof. Then for every $T \subseteq C^+$

$$\mathcal{C}^+ \models \chi(T) \iff T \text{ contains the output node or } T = \emptyset,$$

where

$$\chi(Z) := \forall x \forall y((\mathit{OUT}\, x \wedge \neg Zx) \rightarrow \neg Zy).$$

Recall the formulas *posisat*(Z) and *con*(Z) of Lemma 22 and of the preceding proof, respectively. We set

$$\varphi(Z) := \big(\chi(Z) \wedge \mathit{posisat}(Z) \wedge \mathit{cons}(Z)\big).$$

Clearly, $\varphi(Z) \in \Pi_{1/3}$ up to logical equivalence. We show that the following holds.

$$\mathcal{C} \text{ is not satisfiable} \quad \Longleftrightarrow \quad (\mathcal{C}^+, 0) \in p\text{-MAXIMAL-WD}_\varphi. \tag{20}$$

Observe that $\emptyset$, the empty set, is a solution of $\varphi(Z)$ in $\mathcal{C}^+$ of size 0. Now assume $\mathcal{C}$ is not satisfiable. If $\emptyset$ is not a maximal solution, then there exists some $S \neq \emptyset$ with $\mathcal{C}^+ \models \varphi(S)$. By $\mathcal{C}^+ \models \chi(S)$, the set S has to contain the output node. Since $\mathcal{C}^+ \models \mathit{posisat}(S)$, by Lemma 22 we see that $\mathcal{C}^+$ is satisfiable. But then $\mathcal{C}$ is satisfiable by $\mathcal{C}^+ \models \mathit{cons}(S)$ and (11), a contradiction.

For the converse direction of (20), by contradiction we assume that S is a satisfying assignment of $\mathcal{C}$. Then by Lemma 21 and Lemma 22 there exists some $T \subseteq C^+$ containing the output node such that

$$\mathcal{C}^+ \models (\mathit{cons} \wedge \mathit{posisat})\big(T\big).$$

Obviously

$$\mathcal{C}^+ \models \chi\big(T\big).$$

Altogether, T is a solution of φ in $\mathcal{C}^+$ extending $\emptyset$. Hence, $\emptyset$ is not a maximal solution of φ in $\mathcal{C}^+$.

By (20) the classical "circuit unsatisfiability" problem is reducible in polynomial time to the 0th slice P_0 of p-MAXIMAL-WD$_\varphi$, where for $\ell \in \mathbb{N}$ the ℓth slice P_ℓ of p-MAXIMAL-WD$_\varphi$ is defined by

$$P_\ell := \big\{(\mathcal{A}, \ell) \mid (\mathcal{A}, \ell) \in p\text{-MAXIMAL-WD}_\varphi\big\}.$$

Since the circuit unsatisfiability problem is co-NP-hard, the problem p-MAXIMAL-WD$_\varphi$ is hard for para-(co-NP) under fpt-reductions by Proposition 14 of [15]. □

Corollary 25. *p-MAXIMAL-WSAT$(\Gamma_{1,3})$ is hard for* para-(co-NP) *under fpt-reductions.*

So we have seen that there is a $\Pi_{1/3}$-formula $\varphi(Z)$ such that p-MAXIMAL-WD$_\varphi$ is hard for para-co-NP under fpt-reductions and hence, para-NP-hard under fpt Turing reductions. On the other hand we have:

Proposition 26. *Assume* NP $\neq$ co-NP. *Let $t \geq 1$ and $\varphi(Z)$ a Π_t-formula. Then p-MAXIMAL-WD$_\varphi$ is not* para-NP-*hard under fpt-reductions.*

Proof: For contradiction assume that p-MAXIMAL-WD$_\varphi$ is para-NP-hard under fpt-reductions for some Π_t-formula. For $\ell \in \mathbb{N}$ denote again by P_ℓ the ℓth slice of p-MAXIMAL-WD$_\varphi$ (for its definition see the proof of the previous proposition). Then, by Proposition 15 of [15], there is $d \geq 0$ such that the (classical) problem $P^* := P_0 \cup \ldots \cup P_d$ is NP-hard under PTIME-reductions. But the complement of every slice P_ℓ is in NP, as shown by the following algorithm: Given an instance $(\mathcal{A}, k)$, if $k \neq \ell$ accept and if $k = \ell$, then for every subset $S \subseteq A$ of size ℓ, check whether $\mathcal{A} \models \varphi(S)$ and in the positive case guess a subset S' with $S \subset S' \subseteq A$ and $\mathcal{A} \models \varphi(S')$. Therefore P^* is in co-NP and hence NP $\subseteq$ co-NP, contradicting our assumption. □

6. Maximality problems for negative formulas

Various problems such as p-INDEPENDENT-SET, p-SET-PACKING, or p-RED/BLUE-NONBLOCKER are Fagin-definable by formulas $\varphi(Z)$ negative in Z. The antimonotonicity of formulas $\varphi(Z)$ negative in Z allows to bound the complexity of p-MAXIMAL-WD$_\varphi$ as claimed in Theorem 2. We turn to a proof of this theorem.

As already mentioned in the Introduction, for $\varphi(Z)$ negative in Z, by antimonotonicity, a solution of size k is a maximal solution if no superset of it of size $k+1$ is a solution; more precisely:

Lemma 27. *Let $\varphi(Z)$ be negative in Z and S a solution of φ in the structure $\mathcal{A}$. Then the following are equivalent:*

(i) *S is a maximal solution of φ in $\mathcal{A}$.*

(ii) *For all $b \in A \setminus S$ the set $S \cup \{b\}$ is not a solution of φ in $\mathcal{A}$.*

For a formula $\varphi(Z)$ let

$$\textit{1-max-}\varphi(Z) := \varphi(Z) \wedge \forall y\big(Zy \vee \neg\varphi(Z \cup \{y\})\big), \tag{21}$$

where $\varphi(Z \cup \{y\})$ denotes the formula obtained from φ by replacing atomic formulas Zx by $(Zx \vee x = y)$. By the previous lemma the proof of the following one is straightforward.

Lemma 28. *Let $\varphi(Z)$ be negative in Z.*

(a) p-MAXIMAL-WD$_\varphi$ $=$ p-WD$_{\textit{1-max-}\varphi}$. *Moreover, the maximal solutions of $\varphi(Z)$ coincide with the solutions of 1-max-$\varphi(Z)$.*

(b) *Let $t \geq 1$. If $\varphi(Z)$ is a Π_t-formula, then 1-max-φ is (equivalent to) a Π_{t+1}-formula.*

Using these observations we already get a part of Theorem 2:

Lemma 29. *For $t \geq 1$ and every Π_t-formula $\varphi(Z)$ negative in Z we have* p-MAXIMAL-WD$_\varphi \in$ W$[t+1]$.

Proof: Let $t \geq 1$ and $\varphi(Z)$ be a Π_t-formula negative in Z. Since *1-max-*φ is a Π_{t+1}-formula, the problem p-WD$_{\textit{1-max-}\varphi}$ ($= p$-MAXIMAL-WD$_\varphi$) is in W$[t+1]$ by Theorem 1 (a). □

We turn to a proof of the remaining claims of Theorem 2. We need the following well-known variant of Theorem 4 (c) (for a proof see [19]).

For a set Γ of propositional formulas, we consider the *parameterized partitioned satisfiability problem*

p-PSAT$(\Gamma_{t,1})$	
Input:	A formula $\alpha \in \Gamma_{t,1}$ and a partition $\mathcal{X}_1, \ldots, \mathcal{X}_k$ of the set Var(α) of variables of α.
Parameter:	k (the number of sets in the partition).
Question:	Decide whether $(\alpha, \mathcal{X}_1, \ldots, \mathcal{X}_k)$ is *satisfiable*, that is, whether α has a satisfying assignment that sets exactly one variable in each $\mathcal{X}_\ell$ to TRUE.

Lemma 30. *Let $t \geq 2$. Then:*

(a) p-PSAT$(\Gamma_{t,1})$ *is* W$[t]$*-complete.*

(b) *For even t, the problem* p-PSAT$(\Gamma^+_{t,1})$ *is* W$[t]$*-complete.*

We abbreviate formulas of the form $\exists x(Zx \wedge \psi)$ and $\forall x(Zx \rightarrow \psi)$ by $(\exists x \in Z)\psi$ and $(\forall x \in Z)\psi$, respectively. Note that in the first formula the displayed occurrence of Z is positive while it is negative in the second one.

The next proposition contains the main step of the proof of Theorem 2. Since its proof is quite involved, let us first explain the underlying idea. The proposition claims that there is a $\Pi_{t/2}$-formula $\varphi(Z)$ negative in Z such that p-MAXIMAL-WD$_\varphi$ is W$[t+1]$-complete.

Let us take $t = 3$. A "typical $\Pi_{3/2}$-formula $\varphi(Z)$ negative in Z" has the form

$$\forall u_1 \exists u_2 (\forall z_1 \in Z)(\forall z_2 \in Z)\chi(u_1, u_2, z_1, z_2)$$

with quantifier-free χ (by the way the expression "typical $\Pi_{3/2}$-formula $\varphi(Z)$ negative in Z" can be made precise by the notion of generic formula; we introduce these generic formulas after the proof of the proposition). Then p-MAXIMAL-WD$_\varphi$ is Fagin-defined by *1-max-*$\varphi = \varphi(Z) \wedge \varphi'(Z)$, where by (21) up to logical equivalence

$$\varphi'(Z) = \forall w \notin Z \exists u_1 \forall u_2 (\exists z_1 \in Z \cup \{w\})(\exists z_2 \in Z \cup \{w\})\neg\chi(u_1, u_2, z_1, z_2).$$

At first glance, $\varphi'(Z)$ looks like the typical $\Pi_{4/2}$-formula positive in Z (so that p-WD$_{\varphi'}$ would be W$[4]$-complete by Theorem 1 (a)). However note that the variable w does not occur in χ. To remedy this, χ has to be chosen in such a way that one is forced to take w as variable z_1 and $z_2 \in Z$; then the formula $\varphi'(Z)$ essentially is equivalent to

$$\forall w \exists u_1 \forall u_2 \exists z_2 \in Z \neg\chi(u_1, u_2, w, z_2),$$

which is the typical $\Pi_{4/1}$-formula positive in Z; hence the problem Fagin-defined by it is W$[4]$-complete by Theorem 1 (a).

Proposition 31. *Let $t \geq 1$ be odd. Then there is a $\Pi_{t/2}$-formula $\varphi(Z)$ negative in Z such that the problem p-MAXIMAL-WD$_\varphi$ is W$[t+1]$-hard under fpt-reductions.*

Proof: For notational simplicity let $t = 3$. We show that the W$[4]$-complete problem p-PSAT$(\Gamma^+_{4,1})$ is reducible to p-MAXIMAL-WD$_\varphi$ for some $\varphi(Z)$ of the desired form.

Let $(\alpha, \mathcal{X}_1, \ldots, \mathcal{X}_k)$ be an instance of p-PSAT$(\Gamma^+_{4,1})$. Let $\tau_{\text{parse}} := \{E, \mathit{FIR}, \mathit{VAR}\}$ and consider the parse structure $\mathcal{A}_{\text{parse}}(\alpha)$ of α (cf. Subsection 2.3). We set $\tau := \tau_{\text{parse}} \cup \{\mathit{DIFF}\}$ with a binary relation symbol *DIFF*. Let $\mathcal{A}_1$ $(= \mathcal{A}_1(\alpha))$ be the τ-expansion of $\mathcal{A}_{\text{parse}}(\alpha)$ obtained by setting

$$\mathit{DIFF}^{\mathcal{A}_1} := \big\{(X, X') \mid \text{there are } i, i' \in [k] \text{ with } i \neq i' \text{ and } X \in \mathcal{X}_i \text{ and } X' \in \mathcal{X}_{i'}\big\}.$$

To get the τ-structure $\mathcal{A}$ $(= \mathcal{A}(\alpha))$ we replace in $\mathcal{A}_1$ every node $w \in \mathit{FIR}^{\mathcal{A}_1}$ by k many copies $(w, 1), \ldots,$ (w, k), that is, we set

$$A := (\mathit{FIR}^{\mathcal{A}_1} \times [k]) \cup (A_1 \setminus \mathit{FIR}^{\mathcal{A}_1})$$

and define the relations in $\mathcal{A}$ in such a way that the projection $\pi : A \to A_1$ defined by

$$\pi(a) := \begin{cases} a & \text{if } a \in A_1 \setminus \mathit{FIR}^{\mathcal{A}_1} \\ w & \text{if } a = (w, i) \text{ for some } w \in \mathit{FIR}^{\mathcal{A}_1} \text{ and } i \in [k] \end{cases}$$

is a strong homomorphism from $\mathcal{A}$ to $\mathcal{A}_1$, that is, for all $a, b \in A$ we have

$$\big(\mathit{DIFF}^{\mathcal{A}} ab \iff \mathit{DIFF}^{\mathcal{A}_1} \pi(a)\pi(b)\big), \;\; \big(\mathit{FIR}^{\mathcal{A}} a \iff \mathit{FIR}^{\mathcal{A}_1} \pi(a)\big), \;\; \big(\mathit{VAR}^{\mathcal{A}} a \iff \mathit{VAR}^{\mathcal{A}_1} \pi(a)\big).$$

We let

$$\theta(x_1, x_2, x_3, x_4) := \big(\mathit{FIR}\; x_1 \to (Ex_1x_2 \wedge (Ex_2x_3 \to Ex_3x_4))\big). \tag{22}$$

Clearly, for all $(w, i), (w, j) \in \mathit{FIR}^{\mathcal{A}}$ and all $a, b, c \in A$, we have

$$\mathcal{A} \models \theta((w, i), a, b, c) \quad \iff \quad \mathcal{A} \models \theta((w, j), a, b, c). \tag{23}$$

Moreover, for $S \subseteq \text{Var}(\alpha)$ one easily verifies

$$S \text{ satisfies } \alpha \quad \iff \quad \mathcal{A} \models \forall x_1 \exists x_2 \forall x_3 (\exists x_4 \in S)\; \theta(x_1, x_2, x_3, x_4). \tag{24}$$

We set

$$\varphi(Z) := \forall u_1 \exists u_2 (\forall z_1 \in Z)(\forall z_2 \in Z)\Big(\underbrace{((\mathit{VAR}\, z_1 \wedge \mathit{VAR}\, z_2) \to (z_1 = z_2 \vee \mathit{DIFF}\, z_1 z_2))}_{\chi_1} \wedge \underbrace{((\neg\mathit{VAR}\, z_1 \wedge \mathit{VAR}\, z_2) \to \neg\theta(z_1, u_1, u_2, z_2))}_{\chi_2} \Big)$$

and show that

$$(\alpha, \mathcal{X}_1, \ldots, \mathcal{X}_k) \text{ is satisfiable} \iff (\mathcal{A}, k) \in p\text{-MAXIMAL-WD}_\varphi.$$

Since $\varphi(Z)$ is a $\Pi_{3/2}$-formula negative in Z, this yields our claim. Note that p-MAXIMAL-WD$_\varphi$ = p-WD$_{\textit{1-max-}\varphi}$, where (compare (21)) *1-max-*$\varphi = \varphi(Z) \wedge \varphi'(Z)$ with

$$\varphi'(Z) = \forall w \notin Z \underbrace{\exists u_1 \forall u_2 (\exists z_1 \in Z \cup \{w\})(\exists z_2 \in Z \cup \{w\})(\neg\chi_1 \vee \neg\chi_2)}_{\varphi''(w,Z)}.$$

Clearly (up to logical equivalence)

$$\neg\chi_1 = (\mathit{VAR}\, z_1 \wedge \mathit{VAR}\, z_2 \wedge \neg z_1 = z_2 \wedge \neg \mathit{DIFF}\, z_1 z_2)$$
$$\neg\chi_2 = (\neg\mathit{VAR}\, z_1 \wedge \mathit{VAR}\, z_2 \wedge \theta(z_1, u_1, u_2, z_2)).$$

We have to show that

$$(\alpha, \mathcal{X}_1, \dots, \mathcal{X}_k) \text{ is satisfiable } \iff (\mathcal{A}, k) \in p\text{-WD}_{\textit{1-max-}\varphi},$$

which (by Lemma 28) is our claim. In the following, for easier reading, we mostly denote the interpretation of a variable in $\mathcal{A}$ by the variable itself.

First let S witness that $(\alpha, \mathcal{X}_1, \dots, \mathcal{X}_k)$ is satisfiable. Then $|S| = k$ and $z_1 = z_2 \vee \mathit{DIFF}^{\mathcal{A}}\, z_1 z_2$ for all $z_1, z_2 \in S$ and hence

$$\mathcal{A} \models \varphi(S).$$

We show that $\mathcal{A} \models \varphi'(S)$. Choose $w \notin S$. If w is a variable, say $w \in \mathcal{X}_i$, then set $z_1 := w$ and let z_2 be the variable in $S \cap \mathcal{X}_i$. Then $\mathcal{A} \models (\mathit{VAR}\, z_1 \wedge \mathit{VAR}\, z_2 \wedge \neg z_1 = z_2 \wedge \neg\mathit{DIFF}\, z_1 z_2)$ and hence $\mathcal{A} \models \varphi''(w, S)$. Now assume that w is not a variable. Since S satisfies α, by (24) we know that

$$\mathcal{A} \models \exists u_1 \forall u_2 (\exists z_2 \in S)(\neg\mathit{VAR}\, w \wedge \mathit{VAR}\, z_2 \wedge \theta(w, u_1, u_2, z_2))$$

and hence $\mathcal{A} \models \varphi''(w, S)$. Altogether, $\mathcal{A} \models \varphi'(S)$.

Assume now that $|S| = k$ and $\mathcal{A} \models$ *1-max-*$\varphi(S)$. Choose $w \notin S$ which is not a variable. Since $\mathcal{A} \models \varphi'(S)$, we see that $S \cap \mathrm{Var}(\alpha)$ is nonempty. Then $\mathcal{A} \models \varphi(S)$ shows that $S \cap \mathrm{Var}(\alpha) = \{X_1, \dots, X_\ell\}$ for some $\ell \in [k]$, and $X_j \in \mathcal{X}_{i_j}$ for $j \in [\ell]$ and $1 \le i_1 < \ldots < i_\ell < k$.

Claim: $\ell = k$ and hence $S \cap \mathrm{Var}(\alpha) = S$.

Proof: By contradiction assume that $a_0 \in S \setminus \mathrm{Var}(\alpha)$. Since $\mathcal{A} \models \varphi(S)$, we have

$$\mathcal{A} \models \forall u_1 \exists u_2 (\forall z_1 \in S)(\forall z_2 \in S)\Big((\chi_1 \wedge \chi_2) \wedge (\mathit{VAR}\, z_2 \to \neg\theta(a_0, u_1, u_2, z_2))\Big). \tag{25}$$

In particular (cf. (22)), $\mathit{FIR}^{\mathcal{A}}\, a_0$, say, $a_0 = (w, j)$. Choose $i \in [k]$ such that $(w, i) \notin S$ (such an i exists, as S contains k elements, at least one being a variable). Since $\mathcal{A} \models \varphi'(S)$, that is,

$$\mathcal{A} \models \forall w \notin S \exists u_1 \forall u_2 (\exists z_1 \in S \cup \{w\})(\exists z_2 \in S \cup \{w\})(\neg\chi_1 \vee \neg\chi_2), \tag{26}$$

choose for "$(w, i) \notin S$" a u_1 such that "the rest of this formula" holds. By (25), for this u_1 there is a u_2 such that the rest of the formula in (25) holds. Now, for these u_1, u_2, we find $z_1, z_2 \in S \cup \{(w, i)\}$ such that the rest of the formula in (26) holds. Then not both, z_1 and z_2 can be in S; otherwise this would contradict (25) and our choice of u_2. Therefore $z_1 = (w, i)$ and $\mathcal{A} \models \theta(z_1, u_1, u_2, z_2)$. But then $\mathcal{A} \models \theta((w, i), u_1, u_2, z_2)$ and $\mathcal{A} \models \neg\theta((w, j), u_1, u_2, z_2)$ which contradicts (23). This finishes the proof of the claim.

So we know that $S = \{X_1, \dots, X_k\}$ with $X_i \in \mathcal{X}_i$ for $i \in [k]$. Hence

$$\mathcal{A} \models (\forall z_1 \in S)(\forall z_2 \in S)(z_1 = z_2 \vee \mathit{DIFF}\, z_1 z_2). \tag{27}$$

It remains to show that S satisfies α or equivalently (by (24)), that

$$\mathcal{A} \models \forall x_1 \underbrace{\exists x_2 \forall x_3 (\exists x_4 \in S)\ \theta(x_1, x_2, x_3, x_4)}_{\psi(x_1, S)}.$$

Let $x_1 \in A$. If not $\mathit{Fir}^{\mathcal{A}}\, x_1$, then $\mathcal{A} \models \forall x_2 \forall x_3 \forall x_4\, \theta(x_1, x_2, x_3, x_4)$ and therefore $\mathcal{A} \models \psi(x_1, S)$. So assume $\mathit{Fir}^{\mathcal{A}}\, x_1$. By (27) we have $\mathcal{A} \models (\forall z_1 \in S \cup \{x_1\})(\forall z_2 \in S \cup \{x_1\})\chi_1$. Since $\mathcal{A} \models \varphi'(S)$, we get $\mathcal{A} \models \exists u_1 \forall u_2 (\exists z_1 \in S \cup \{x_1\})(\exists z_2 \in S \cup \{x_1\}) \neg \chi_2$ and therefore $\mathcal{A} \models \exists u_1 \forall u_2 (\exists z_2 \in S) \theta(x_1, u_1, u_2, z_2))$. This just says that $\mathcal{A} \models \psi(x_1, S)$. Altogether, $\mathcal{A} \models \forall x_1 \psi(x_1, S)$. □

Let $t, d \geq 1$ and fix a $(t-1+d)$-ary relation symbol $R = R_{t,d}$. For odd $t \geq 1$ the *generic* formula $\mathit{gen}[t,d](Z)$ is defined by

$$\mathit{gen}[t,d](Z) := \forall y_1 \exists y_2 \forall y_3 \ldots \exists y_{t-1} (\forall z_1 \in Z) \ldots (\forall z_d \in Z)\ R y_1 \ldots y_{t-1} z_1 \ldots z_d.$$

Note that $\mathit{gen}[t,d](Z)$ is (equivalent) to a $\Pi_{t/d}$-formula negative in Z. It is known (cf. [19, 18]):

Theorem 32. *Let $t, d \geq 1$ with $t + d \geq 3$ and odd t. Then p-$\mathrm{WD}_{\mathit{gen}[t,d]}$ is $\mathrm{W}[t]$-complete under* fpt-*reductions.*

For every formula $\varphi(Z)$ of the form

$$\forall y_1 \exists y_2 \forall y_3 \ldots \exists y_{t-1} (\forall z_1 \in Z) \ldots (\forall z_d \in Z)\ \psi,$$

where ψ is quantifier-free and does not contain Z, one easily verifies that p-MAXIMAL-$\mathrm{WD}_{\varphi} \leq^{\mathrm{fpt}}$ p-MAXIMAL-$\mathrm{WD}_{\mathit{gen}[t,d]}$. As the formula $\varphi(Z)$ in the proof of Proposition 31 has this form for $d = 2$, we get:

Corollary 33. *For every $d \geq 2$ and every odd $t \geq 1$ the problem p-MAXIMAL-$\mathrm{WD}_{\mathit{gen}[t,d]}$ is $\mathrm{W}[t+1]$-complete under fpt-reductions.*

In terms of weighted satisfiability problems, by Corollary 17 (c) we have shown:

Corollary 34. *For every $d \geq 2$ and every odd $t \geq 1$ the problem p-MAXIMAL-WSAT$(\Gamma^-_{t,d})$ is $\mathrm{W}[t+1]$-complete under fpt-reductions.*

In the proof of next result we need the following lemma, implicitly containps ed in the proof of Theorem 7.29 in [18], the Monotone and Antimonotone Collapse Theorem:

Lemma 35. *Let $t \geq 2$ and $d \geq 1$. Then there is an fpt-algorithm associating with every instance (α, k) of p-WSAT$(\Gamma^-_{t,d})$ if t is even, and with every instance (α, k) of p-WSAT$(\Gamma^+_{t,d})$ if t is odd, a structure $\mathcal{A}$ (depending on α and k), and a Π_{t-2}-formula $\psi_{t-2}(y, x_1, \ldots, x_k)$, (depending on k but not on α), with quantifier blocks of length 1 such that:*

- *If $\{X_1, \ldots, X_n\}$ is the set of variables of α, then $[n] \subseteq A$.*
- *The vocabulary of the structure $\mathcal{A}$ is of arity $\leq d+2$ (the arity of a vocabulary being the maximum of the arities of its symbols); it contains unary relation symbols VAR and ROOT and $\mathit{VAR}^{\mathcal{A}} = [n]$ and $\mathit{ROOT}^{\mathcal{A}}$ is a singleton.*
- *For $a \in \mathit{ROOT}^{\mathcal{A}}$ and arbitrary $m_1, \ldots, m_k \in [n]$ we have:*

$$\{X_{m_1}, \ldots, X_{m_k}\} \textit{ satisfies } \alpha \iff \mathcal{A} \models \psi_{t-2}(a, m_1, \ldots, m_k).$$

Lemma 36. *Let $t \geq 2$ be even and $d \geq 1$. Then p-MAXIMAL-WSAT$(\Gamma^-_{t,d}) \in \mathrm{W}[t]$.*

Proof: Let (α, k) be an instance of p-MAXIMAL-WSAT$(\Gamma^-_{t,d})$. For the instance $(\alpha, k+1)$ of p-WSAT$(\Gamma^-_{t,d})$ choose the structure $\mathcal{A}$ and the formula $\psi_{t-2}(y, x_1, \ldots, x_k, x_{k+1})$ according to the preceding lemma. Then, by this lemma, $(\alpha, k) \in p$-MAXIMAL-WSAT$(\Gamma^-_{t,d})$ if and only if

$$\mathcal{A} \models \exists x_1 \ldots \exists x_k \exists y \Big(\big(\bigwedge_{i \in [k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j \in [k] \\ i \neq j}} x_i \neq x_j \wedge \mathit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \ldots, x_k, x_k) \big) \wedge \forall x_{k+1} \big((\mathit{VAR}\, x_{k+1} \wedge \bigwedge_{i \in [k]} x_i \neq x_{k+1}) \rightarrow \neg \psi_{t-2}(y, x_1, \ldots, x_k, x_{k+1}) \big) \Big).$$

Since this formula is equivalent to a $\Sigma_{t,2}$-sentence, we thus have shown that p-MAXIMAL-WSAT$(\Gamma^-_{t,d})$ is fpt-reducible to the model-checking problem p-MC$(\Sigma_{t,2})$ and hence it is in W$[t]$ by Theorem 4 (a). □

Proof of Theorem 2: Let $t \geq 1$ be odd. Then

$$\begin{aligned} \mathrm{W}[t+1] &\subseteq \big[\{p\text{-MAXIMAL-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula negative in } Z\}\big]^{\text{fpt}} && \text{(by Proposition 31)} \\ &\subseteq \big[\{p\text{-MAXIMAL-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_{t+1}\text{-formula negative in } Z\}\big]^{\text{fpt}} \\ &\subseteq \mathrm{W}[t+1] && \text{(by Lemma 36).} \end{aligned}$$

□

In view of Theorem 2 one might conjecture:

If $\varphi(Z)$ is a Π_1-formula negative in Z and p-WD$_\varphi \in$ FPT, then p-MAXIMAL-WD$_\varphi \in$ W[1].

We disprove this conjecture (unless W[1] = W[2]).

Theorem 37. *The problem*

p-CLIQUE-OR-INDEPENDENT-SET	
Input:	A graph G and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Does G have a clique of size k or an independent set of size k?

is fixed-parameter tractable and p-MAXIMAL-CLIQUE-OR-INDEPENDENT-SET *is* W[2]*-complete. Moreover,* p-CLIQUE-OR-INDEPENDENT-SET *coincides with* p-WD$_\varphi$ *for some* Π_1*-formula* $\varphi(Z)$.

Proof: We start with the last statement. p-CLIQUE-OR-INDEPENDENT-SET is p-WD$_\varphi$ for a Π_1-formula $\varphi(Z)$ equivalent to the conjunction $\forall x \neg Exx \wedge \forall x \forall y(Exy \rightarrow Eyx)$ of the axioms for graphs with the formula

$$(\forall x \in Z)(\forall y \in Z)(\forall u \in Z)(\forall v \in Z)\Big((x = y \vee Exy) \vee \neg Euv\Big).$$

By Theorem 2 this shows that p-MAXIMAL-CLIQUE-OR-INDEPENDENT-SET $\in$ W[2].

p-MAXIMAL-CLIQUE-OR-INDEPENDENT-SET is W[2]-hard: For a graph $\mathcal{G}$ and $k \geq 1$ let $\mathcal{G}[k]$ be the graph obtained from $\mathcal{G}$ by adding $k+1$ isolated vertices. For $k \geq 2$ we have

$$(\mathcal{G}, k) \in p\text{-MAXIMAL-CLIQUE} \iff (\mathcal{G}[k], k) \in p\text{-MAXIMAL-CLIQUE-OR-INDEPENDENT-SET}.$$

By Proposition 9 this yields the claim.

p-CLIQUE-OR-INDEPENDENT-SET is fixed-parameter tractable: By Ramsey's Theorem if the graph G has at least 2^{2k} many vertices, then it either contains a clique of size k or an independent set of size k. □

The fixed-parameter tractability of p-CLIQUE-OR-INDEPENDENT-SET is from [24].

6.1. Non-maximal solutions of negative formulas. Contrary to the maximality problems, the non-maximality problems do not increase the complexity for formulas $\varphi(Z)$ negative in Z. In fact, for such a formula $\varphi(Z)$, by antimonotonicity we have for all structures $\mathcal{A}$ and every $k \geq 1$

$$(\mathcal{A}, k) \in p\text{-WD}_\varphi \iff (\mathcal{A}, k-1) \in p\text{-NON-MAXIMAL-WD}_\varphi, \tag{28}$$

and hence

$$p\text{-WD}_\varphi \equiv^{\text{fpt}} p\text{-NON-MAXIMAL-WD}_\varphi.$$

By Theorem 1 (b), we therefore get:

Theorem 38. *For odd $t \geq 1$,*

$$\begin{aligned} \mathrm{W}[t] &= \big[\{p\text{-Non-Maximal-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula negative in } Z\}\big]^{\mathrm{fpt}} \\ &= \big[\{p\text{-Non-Maximal-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_{t+1}\text{-formula negative in } Z\}\big]^{\mathrm{fpt}}. \end{aligned}$$

Remark 39. It is not hard to show:

- There is a Π_1-formula $\varphi(Z)$ such that the problem p-NON-MAXIMAL-WD$_\varphi$ is para-NP-hard under fpt-reductions.
- Unless NP $=$ co-NP, for all $t \geq 1$ and every Π_t-formula the problem p-NON-MAXIMAL-WD$_\varphi$ is not para-(co-NP)-hard under fpt-reductions.

7. Minimality problems

Recall (from the Introduction and Section 4) that

- p-MINIMAL-WD$_\varphi$ asks for minimal solutions of a given size,
- p-MINIMAL-WD$_\varphi$ is trivial for formulas $\varphi(Z)$ negative in Z.

Contrary to the case of maximality problems where we saw that p-MAXIMAL-WD$_\varphi$ could be of very high complexity already for $\varphi \in \Pi_1$, the situation is quite different for minimality problems, as shown by Theorem 3 which we prove in this section. In view of Corollary 17 we get claim (c) of this theorem, namely

$$p\text{-Minimal-WD}_\varphi \in \text{FPT for every } \Pi_1\text{-formula } \varphi(Z),$$

by Theorem 10.

We show the remaining claims (a) and (b) of Theorem 3 in the terminology of weighted satisfiability problems. In fact they correspond to Lemma 41, Lemma 42, and Lemma 43.

The proof of the following well-known result is simple:

Lemma 40. *Let $s \geq 1$ and $\mathcal{A}$ a structure containing among others unary relations $R_1^{\mathcal{A}}, \dots, R_s^{\mathcal{A}}$, which are pairwise disjoint and nonempty. Furthermore let $\varphi_i(\bar{x})$ for $i \in [s]$ be formulas of first-order logic, $Q \in \{\forall, \exists\}$ and u, y distinct variables with u not contained in $\bar{x}$. Then:*

(a) $\mathcal{A} \models \forall \bar{x}\Big(\bigwedge_{i\in[s]} \varphi_i \longleftrightarrow \forall u \bigwedge_{i\in[s]}(R_i u \to \varphi_i)\Big)$;

(b) $\mathcal{A} \models \forall \bar{x}\Big(\bigvee_{i\in[s]} \varphi_i \longleftrightarrow \exists u \bigvee_{i\in[s]}(R_i u \wedge \varphi_i)\Big)$;

(c) $\mathcal{A} \models \forall \bar{x}\forall u\Big(\bigwedge_{i\in[s]}(R_i u \to Qy\varphi_i) \longleftrightarrow Qy \bigwedge_{i\in[s]}(R_i u \to \varphi_i)\Big)$;

(d) $\mathcal{A} \models \forall \bar{x}\forall u\Big(\bigvee_{i\in[s]}(R_i u \wedge Qy\varphi_i) \longleftrightarrow Qy \bigvee_{i\in[s]}(R_i u \wedge \varphi_i)\Big)$.

In the following proofs we often tacitly use the fact that for α in some $\Gamma^+_{t,d}$ and every assignment S satisfying α, we have:

$$S \text{ is a minimal satisfying assignment of } \alpha \iff \text{for all } X \in S\text{: } S \setminus \{X\} \text{ does not satisfy } \alpha.$$

Lemma 41. *Let $t > 1$ be odd and $d \geq 1$. Then p-MINIMAL-WSAT$(\Gamma^+_{t,d}) \in$ W$[t-1]$.*

Proof: We show that p-MINIMAL-WSAT$(\Gamma^+_{t,d}) \leq p$-MC$(\Sigma_{t-1,2})$, which yields the claim by Theorem 4 (a). We use Lemma 35 and its terminology. Let (α, k) be an instance of p-MINIMAL-WSAT$(\Gamma^+_{t,d})$

and hence of p-WSAT$(\Gamma^+_{t,d})$. For the corresponding structure $\mathcal{A}$ and the Π_{t-2}-formula ψ_{t-2}, we have $(\alpha, k) \in p$-MINIMAL-WSAT$(\Gamma^+_{t,d})$ if and only if

$$\mathcal{A} \models \exists x_1 \dots \exists x_k \exists y \Big(\bigwedge_{i\in[k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j\in[k]\\ i\neq j}} x_i \neq x_j \wedge \mathit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \dots, x_k) \wedge$$
$$\bigwedge_{i\in[k]} \neg\psi_{t-2}(y, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, x_{i+1}) \Big),$$

(where $x_{i+1} = x_1$ for $i = k$). Let $\psi_{t-2}(y, x_1, \dots, x_k) = \forall u \psi'(u, y, x_1, \dots, x_k)$. Then the preceding formula is equivalent to

$$\exists x_1 \dots \exists x_k \exists y \exists u_1 \dots \exists u_k \Big(\bigwedge_{i\in[k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j\in[k]\\ i\neq j}} x_i \neq x_j \wedge \mathit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \dots, x_k)$$
$$\wedge \bigwedge_{i\in[k]} \neg\psi'(u_i, y, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, x_{i+1}) \Big).$$

If $t = 3$, then the formula ψ' is quantifier-free and the preceding formula is equivalent to a $\Sigma_{t-1,1}$-formula. Assume $t > 3$. Let $R_1^{\mathcal{A}}, \dots, R_k^{\mathcal{A}}$ be a partition of A into nonempty sets. By Lemma 40 (a), $(\alpha, k) \in p$-MINIMAL-WSAT$(\Gamma^+_{t,d})$ is equivalent to

$$(\mathcal{A}, R_1^{\mathcal{A}}, \dots, R_k^{\mathcal{A}}) \models \exists x_1 \dots \exists x_k \exists y \exists u_1 \dots \exists u_k \forall v$$
$$\Big(\bigwedge_{i\in[k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j\in[k]\\ i\neq j}} x_i \neq x_j \wedge \mathit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \dots, x_k)$$
$$\wedge \bigwedge_{i\in[k]} \big(R_i v \rightarrow \neg\psi'(u_i, y, x_1, \dots, x_{i_1}, x_{i+1}, \dots, x_k, x_{i+1})\big) \Big).$$

Now repeatedly applying Lemma 40 (c) we see that this formula is equivalent in $(\mathcal{A}, R_1^{\mathcal{A}}, \dots, R_k^{\mathcal{A}})$ to a $\Sigma_{t-1,2}$-formula. □

Lemma 42. *Let $t \geq 2$. Then p-*MINIMAL-WSAT$(\Gamma_{t,1})$ *is* W$[t]$*-hard and if t is even, the problem p-*MINIMAL-WSAT$(\Gamma^+_{t,1})$ *is* W$[t]$*-hard.*

Proof: We have p-PSAT$(\Gamma_{t,1}) \leq^{\text{fpt}} p$-MINIMAL-WSAT$(\Gamma_{t,1})$ as witnessed by the reduction

$$(\alpha, \mathcal{X}_1, \dots, \mathcal{X}_k) \mapsto (\alpha \wedge \bigwedge_{i\in[k]} \bigvee_{X\in\mathcal{X}_i} X,\ k).$$

If $\alpha \in \Gamma^+_{t,1}$, then the formula on the right hand side is in $\Gamma^+_{t,1}$, too, so that we get a reduction from p-PSAT$(\Gamma^+_{t,1})$ to p-MINIMAL-WSAT$(\Gamma^+_{t,1})$. Now the claims follow from Lemma 30. □

Lemma 43. *Let $t, d \geq 1$. Then p-*MINIMAL-WSAT$(\Gamma_{t,d}) \in$ W$[t]$.

Proof: So fix $t, d \geq 1$. We show that p-MINIMAL-WSAT$(\Gamma_{t,d}) \leq^{\text{fpt}} p$-MC$(\Sigma_{t,3})$, which proves the claim by Theorem 4 (a).

Using Lemma 6.31 in [18], it is not hard to see that there is an fpt-algorithm associating with every instance (α, k) of p-WSAT$(\Gamma_{t,d})$ a structure $\mathcal{A}$ in a vocabulary τ containing a unary relation symbol *VAR* with $\mathit{VAR}^{\mathcal{A}} = \text{Var}(\alpha)$ and a formula $\varphi(Z)$ such that

(a) $\varphi(Z) = \forall y_1 \exists y_2 \dots Q y_{t-1} \chi$, where χ is a bounded formula and $Q = \forall$ if t is even, and $Q = \exists$ if t is odd (a formula is *bounded* if quantifiers only appear in the form $(\exists x \in Z)\psi$ or in the form $(\forall x \in Z)\psi$);

(b) for all $S \subseteq A$, if $\mathcal{A} \models \varphi(S)$ then $S \subseteq \text{Var}(\alpha)$;

(c) for all $S \subseteq \text{Var}(\alpha)$ with $|S| \le k$

$$\mathcal{A} \models \varphi(S) \quad \Longleftrightarrow \quad S \text{ satisfies } \alpha. \tag{29}$$

(We remark that the formula $\varphi(Z)$ may depend on k, even though it does not depend on α.) For every set V of first-order variables and every bounded formula χ let $\chi[V]$ be the quantifier-free formula obtained from χ by inductively replacing

- atoms Zy by $\bigvee_{x \in V} y = x$
- every quantifier $(\forall y \in Z)\rho(y, \dots)$ by $\bigwedge_{x \in V} \rho(x, \dots)$
- every quantifier $(\exists y \in Z)\rho(y, \dots)$ by $\bigvee_{x \in V} \rho(x, \dots)$.

Let $\varphi[V] := \forall y_1 \exists y_2 \dots Q y_{t-1} \chi[V]$. Then

$$(\alpha, k) \in p\text{-MINIMAL-WSAT}(\Gamma_{t,d}) \iff \mathcal{A} \models \psi,$$

where

$$\psi := \exists x_1 \dots \exists x_k \Big(\bigwedge_{1 \le i < j \le k} x_i \ne x_j \wedge \varphi\big[\{x_1, \dots, x_k\}\big] \wedge \bigwedge_{V \subset \{x_1, \dots, x_k\}} \neg\varphi[V] \Big). \tag{30}$$

For $t = 1$ we thus have a reduction from p-MINIMAL-WSAT$(\Gamma_{1,d})$ to p-MC(Σ_1), showing the claim for $t = 1$. Let $t \ge 2$. Clearly ψ is equivalent to

$$\begin{aligned} \psi' := \exists x_1 \dots \exists x_k (\exists x_V)_{V \subset \{x_1, \dots, x_k\}} \Big(\textstyle\bigwedge_{1 \le i < j \le k} x_i \ne x_j \wedge \varphi[\{x_1, \dots, x_k\}] \wedge \\ \textstyle\bigwedge_{V \subset \{x_1, \dots, x_k\}} \forall y_2 \exists y_3 \dots Q' y_{t-1} \neg \chi(x_V, y_2, \dots, y_{t-1})[V] \Big). \end{aligned} \tag{31}$$

(The formula $\chi(x_V, y_2, \dots, y_{t-1})[V]$ is obtained from χ by first substituting the variable y_1 by x_V and then by replacing the bounded quantifiers as explained above.) Here $Q' = \forall$ if $Q = \exists$ and $Q' = \exists$ if $Q = \forall$. Applying to $\bigwedge_{V \subset \{x_1, \dots, x_k\}} \dots$ transformations according to Lemma 40, we get a $\Sigma_{t,3}$-formula equivalent to ψ' (and hence to ψ) in an expansion of $\mathcal{A}$ by appropriate unary relations. □

We collect what we have shown over minimal weighted satisfiability results in this section:

Theorem 44. *(a)* p-MINIMAL-WSAT$(\Gamma_{t,d})$ *is* W$[t]$*-complete for all* $t \ge 2$ *and* $d \ge 1$.

(b) p-MINIMAL-WSAT$(\Gamma^+_{t,d})$ *is* W$[t]$*-complete for all even* $t \ge 2$ *and* $d \ge 1$.

(c) p-MINIMAL-WSAT$(\Gamma^+_{t,d})$ *is* W$[t-1]$*-complete for all odd* $t \ge 3$ *and* $d \ge 1$.

Proof: Part (a) and part (b) follow by Lemma 43 and Lemma 42, and part (c) by Lemma 41 and since $\Gamma^+_{t-1,d}$ is contained (up to logical equivalence) in $\Gamma^+_{t,d}$. □

Proof of Theorem 3: Immediate by Theorem 10 and the previous theorem and by Corollary 17. □

Remark 45. Let $t, d \ge 1$. Denote by p-MINIMAL$^{\le}$-WSAT$(\Gamma_{t,d})$ the problem asking, given an instance (α, k) of p-WSAT$(\Gamma_{t,d})$, whether there is a minimal solution of size $\le k$. Then p-MINIMAL$^{\le}$-WSAT$(\Gamma_{t,d}) \in$ W$[t]$. This can be shown as Lemma 43. We point out the changes that are necessary. We use the notations of the proof of that lemma. We have

$$(\alpha, k) \in p\text{-MINIMAL}^{\le}\text{-WSAT}(\Gamma_{t,d}) \iff \mathcal{A} \models \psi_{\le},$$

where

$$\psi_{\leq} := \exists x_1 \ldots \exists x_k \bigvee_{0\leq \ell \leq k} \Big(\bigwedge_{1\leq i<j\leq \ell} x_i \neq x_j \wedge \varphi\big[\{x_1,\ldots,x_\ell\}\big] \wedge \bigwedge_{V\subset\{x_1,\ldots,x_\ell\}} \neg\varphi[V]\Big).$$

For $t = 1$ we thus have a reduction from p-MINIMAL$^{\leq}$-WSAT$(\Gamma_{1,d})$ to p-MC(Σ_1), showing the claim for $t = 1$. Let $t \geq 2$. Clearly $\psi_{\leq}$ is equivalent to

$$\psi'_{\leq} := \exists x_1 \ldots \exists x_k (\exists x_V)_{V\subset\{x_1,\ldots,x_k\}} \bigvee\nolimits_{0\leq\ell\leq k} \Big(\bigwedge\nolimits_{1\leq i<j\leq \ell} x_i \neq x_j \wedge \varphi[\{x_1,\ldots,x_\ell\}] \wedge \bigwedge\nolimits_{V\subset\{x_1,\ldots,x_\ell\}} \forall y_2 \exists y_3 \ldots Q' y_{t-1} \neg\chi(x_V, y_2, \ldots, y_{t-1})[V]\Big).$$

By Lemma 40 (c), this formula is equivalent to $\psi''_{\leq}$ in an expansion of $\mathcal{A}$ by appropriate unary relations, where

$$\psi''_{\leq} := \exists x_1 \ldots \exists x_k (\exists x_V)_{V\subset\{x_1,\ldots,x_k\}} \exists u \bigvee\nolimits_{0\leq\ell\leq k} \Big(R_\ell u \wedge \bigwedge\nolimits_{1\leq i<j\leq \ell} x_i \neq x_j \wedge \varphi[\{x_1,\ldots,x_\ell\}] \wedge \bigwedge\nolimits_{V\subset\{x_1,\ldots,x_\ell\}} \forall y_2 \exists y_3 \ldots Q' y_{t-1} \neg\chi(x_V, y_2, \ldots, y_{t-1})[V]\Big),$$

Now one obtains a $\Sigma_{t,3}$-formula by applying to $\psi''_{\leq}$ the same transformations as to (31) in Lemma 43. Note that for $t = 1$ we have the stronger statement p-MINIMAL$^{\leq}$-WSAT$(\Gamma_{1,d}) \in$ FPT, as the algorithm in Theorem 10 lists all satisfying assignments of weight $\leq k$ of a given $\alpha \in \Gamma_{1,d}$.

Example 46. We present an example of a Π_3-formula $\varphi(Z)$ such that

$$p\text{-WD}_\varphi \in \text{FPT} \text{ and } p\text{-MINIMAL-WD}_\varphi \text{ is W[2]-complete;}$$

in particular, p-WD$_\varphi <^{\text{fpt}} p$-MINIMAL-WD$_\varphi$ if FPT $\neq$ W[2]. This explains why, we made the succinct formulation "minimality problems do not increase the complexity in terms of the W-hierarchy" used in the Abstract precise in the Introduction by stating: The quantifier complexity of $\varphi(Z)$ yields the same upper bounds for the complexity of p-MINIMAL-WD$_\varphi$ as for the complexity of p-WD$_\varphi$.

Recall the vocabulary $\tau_{\text{parse}} := \{E, \mathit{FIR}, \mathit{VAR}\}$ and the parse structure $\mathcal{A}_{\text{parse}}(\alpha)$ introduced in Section 2.3 for $\alpha \in \Gamma^+_{2,1}$. Set $\tau := \tau_{\text{parse}} \cup \{\mathit{ROOT}, U\}$ with unary ROOT and U. For $\alpha \in \Gamma^+_{2,1}$ let $\mathcal{A}(\alpha)$ be the τ-structure given by:

$$\begin{aligned} A(\alpha) &= A_{\text{parse}}(\alpha) \,\dot\cup\, \{u\} \\ R^{\mathcal{A}(\alpha)} &= R^{\mathcal{A}_{\text{parse}}(\alpha)} && \text{(for } R \in \tau_{\text{parse}}) \\ \mathit{ROOT}^{\mathcal{A}(\alpha)} &= \{r\} && \text{where } r \text{ denotes the root of } \mathcal{A}_{\text{parse}}(\alpha) \\ U^{\mathcal{A}(\alpha)} &= \{u\}. \end{aligned}$$

It is not difficult to write down a Π_3-formula $\varphi(Z)$ such that for every τ-structure $\mathcal{A}$ and every $S \subseteq A$ we have $\mathcal{A} \models \varphi(S)$ if and only if for some $\alpha \in \Gamma^+_{2,1}$ we have:

(i) $\mathcal{A} = \mathcal{A}(\alpha)$ (up to isomorphism);

(ii) $S \subseteq \mathit{VAR}^{\mathcal{A}}(\alpha) \cup \{u, r\}$;

(iii) $u \in S \iff r \notin S$;

(iv) if $r \in S$, then $S \setminus \{r\}$ satisfies α;

(v) If $u \in S$, then $S \setminus \{u\}$ does not satisfy α.

We have p-WD$_\varphi \in$ FPT: Given an instance $(\mathcal{A}, k)$, first check if, up to isomorphism, $\mathcal{A}$ has the form $\mathcal{A}(\alpha)$ for some $\alpha \in \Gamma^+_{2,1}$. If not reject, otherwise accept if and only if $1 \leq k \leq |\mathit{VAR}^{\mathcal{A}}(\alpha)| + 1$.

We get our second claim as

$$p\text{-MINIMAL-WSAT}(\Gamma^+_{2,1}) \equiv^{\text{fpt}} p\text{-MINIMAL-WD}_\varphi.$$

First we show p-MINIMAL-WSAT$(\Gamma^+_{2,1}) \leq^{\text{fpt}} p$-MINIMAL-WD$_\varphi$: For every instance (α, k) of the problem p-MINIMAL-WSAT$(\Gamma^+_{2,1})$ with $k \geq 1$ we have:

$$(\alpha, k) \in p\text{-MINIMAL-WSAT}(\Gamma^+_{2,1}) \iff (\mathcal{A}(\alpha), k+1) \in p\text{-MINIMAL-WD}_\varphi. \tag{32}$$

In fact, note that for $\alpha \in \Gamma^+_{2,1}$ and $S \subseteq \mathit{VAR}^{\mathcal{A}}(\alpha)$:

$$S \text{ is a minimal assignment satisfying } \alpha \iff S \cup \{r\} \text{ is a minimal solution of } \varphi(Z) \text{ in } \mathcal{A}(\alpha).$$

This gives the implication from left to right in (32). Conversely, assume that T is a minimal solution of $\varphi(Z)$ in $\mathcal{A}(\alpha)$ of size $k+1 \geq 2$. By the previous equivalence, it suffices to show that $T = S \cup \{r\}$ with $S \subseteq \mathit{VAR}^{\mathcal{A}}(\alpha)$. Therefore, by (ii) and (iii), we have to show that $u \notin T$. By contradiction, assume that $u \in T$. Then, by (v), the assignment $T \setminus \{u\}$ does not satisfy α. As $\alpha \in \Gamma^+_{2,1}$ every proper subset of $T \setminus \{u\}$, in particular, the empty set does not satisfy α. Hence, $\mathcal{A}(\alpha) \models \varphi(\{u\})$, contrary to the minimality of T.

Essentially (32) witnesses that p-MINIMAL-WD$_\varphi \leq^{\text{fpt}} p$-MINIMAL-WSAT$(\Gamma^+_{2,1})$. We leave the details to the reader.

7.1. Non-minimal solutions. Besides for $t = 1$, non-minimality problems for Π_t-formulas behave like minimality problems:

Theorem 47. *(a) If $t \geq 1$, then*

$$\text{W}[t] = \big[\{p\text{-NON-MINIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula}\}\big]^{\text{fpt}}.$$

(b) If $t \geq 2$ is even, then

$$\begin{aligned}\text{W}[t] &= \big[\{p\text{-NON-MINIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula positive in } Z\}\big]^{\text{fpt}}\\ &= \big[\{p\text{-NON-MINIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_{t+1}\textit{-formula positive in } Z\}\big]^{\text{fpt}}.\end{aligned}$$

Proof: (a) We have $\text{W}[t] \subseteq \big[\{p\text{-NON-MINIMAL-WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula}\}\big]^{\text{fpt}}$, since for a Π_t-formula $\varphi(Z)$ and $(\mathcal{A}, k)$ with $k \neq 0$

$$(\mathcal{A}, k) \in p\text{-WD}_\varphi \iff (\mathcal{A}, k) \in p\text{-NON-MINIMAL-WD}_{(\varphi \vee \forall x \neg Zx)}.$$

For the reverse inclusion one first convinces oneself using Lemma 15 that it suffices to show that the problem p-NON-MINIMAL-WSAT$(\Gamma_{t,d})$ is in W$[t]$ for $t, d \geq 1$. But this is shown as Lemma 43: Instead of the formula (30), we now have the formula

$$\psi := \exists x_1 \ldots \exists x_k \Big(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \varphi\big[\{x_1, \ldots, x_k\}\big] \wedge \bigvee_{V \subset \{x_1, \ldots x_k\}} \varphi[V] \Big).$$

We turn to (b). For $\varphi(Z)$ positive in Z, we have

$$(\mathcal{A}, k) \in p\text{-WD}_\varphi \iff (\mathcal{A}, k+1) \in p\text{-NON-MINIMAL-WD}_\varphi \tag{33}$$

for all $(\mathcal{A}, k)$ with $|A| \geq k+1$ and hence

$$p\text{-WD}_\varphi \equiv^{\text{fpt}} p\text{-NON-MINIMAL-WD}_\varphi.$$

Therefore we get (b) by Theorem 1 (c). □

8. Construction and listing problems

8.1. Listing, construction and counting problems. Often we are not only interested in knowing whether there is a (maximal/minimal) solution of size k, the "decision problem," but we also want to construct a (maximal/minimal) solution of size k, the "construction problem," or we want to list all (maximal/minimal) solutions of a given size, the "listing problem" or, finally, we want to count the number of (maximal/minimal) solutions, the "counting problem." To be precise let us define the construction problem p-CONSTR-MAXIMAL-WD$_\varphi$, the listing problem p-LIST-MAXIMAL-WD$_\varphi$, and the counting problem p-#MAXIMAL-WD$_\varphi$:

p-CONSTR-MAXIMAL-WD$_\varphi$

Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Construct a maximal solution of $\varphi(Z)$ in $\mathcal{A}$ of size k, if there is one, otherwise reject.

p-LIST-MAXIMAL-WD$_\varphi$

Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	List all maximal solutions of $\varphi(Z)$ in $\mathcal{A}$ of size k.

p-#MAXIMAL-WD$_\varphi$

Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Compute the number of maximal solutions of $\varphi(Z)$ in $\mathcal{A}$ of size k.

It is well-known that for Fagin-definable problems the construction and listing problems are reducible to the corresponding decision problems. This is shown using the "self-reducibility of Fagin-definable problems." Since we are going to refer to a proof of this result, when considering construction and listing for maximality/minimality problems, we present and prove a precise formulation of it (cf. [22], too):

Let $\psi(Z)$ be a formula of vocabulary τ and let Q^+ and Q^- be unary relation symbols not in τ. We set

$$\textit{constr}(Z) := \forall x(Q^+x \to Zx) \quad \text{and} \quad \textit{list}(Z) := \forall x\big((Q^+x \to Zx) \wedge (Q^-x \to \neg Zx)\big).$$

In the stepwise construction of a new solution Z of $\psi(Z)$ the relation Q^+ will contain the elements already identified as elements of Z and Q^- (if present) the elements already identified as elements of the complement of Z.

Theorem 48. *For every $\psi(Z)$ we have:*

(a) *There is a polynomial time algorithm with fpt-queries to an oracle for p-WD$_{(\psi\wedge constr)}$ solving the construction problem p-CONSTR-WD$_\psi$.*[3]
(By an fpt-query to an oracle for p-WD$_{(\psi\wedge constr)}$ we mean a query "$(\mathcal{A}, k) \in p$-WD$_{(\psi\wedge constr)}$?" where k is bounded in terms of the parameter of the input.)

(b) *There is a polynomial time delay algorithm with fpt-queries to an oracle for p-WD$_{(\psi\wedge list)}$ solving p-LIST-WD$_\psi$.*

Proof: The algorithms CONSTR$_\psi$ and LIST$_\psi$ solve the corresponding construction and listing problems in the desired time. To be a bit more precise, let $(\mathcal{A}, k)$ be an instance of p-WD$_\psi$ We assume the universe A of $\mathcal{A}$ is ordered. We denote by a_i the ith element of A in this ordering. The algorithm CONSTR$_\psi$ on input $(\mathcal{A}, \emptyset, k)$ constructs the first (with respect to the induced lexicographical ordering) solution of ψ in $\mathcal{A}$ of size k and rejects if there is no such solution, and the algorithm LIST$_\psi$ on input $(\mathcal{A}, \emptyset, \emptyset, k)$ outputs all such solutions in the lexicographical order.

[3]The reader should note that we solve p-CONSTR-WD$_\psi$ and not p-CONSTR-MAXIMAL-WD$_\psi$, that is, we construct an arbitrary solution of size k, if there is one, and reject otherwise.

$\textsc{Constr}_\psi(\mathcal{A}, (Q^+)^\mathcal{A}, \ell)$
// a τ-structure $\mathcal{A}$ with an ordering, $(Q^+)^\mathcal{A} \subseteq A$, and $\ell \in \mathbb{N}$

if $\big((\mathcal{A}, (Q^+)^\mathcal{A}), \ell\big) \notin p\text{-WD}_{(\psi \wedge \textit{constr})}$ **then** reject
if $|(Q^+)^\mathcal{A}| = \ell$ **then** return $(Q^+)^\mathcal{A}$
$i \leftarrow 1 + \max\{j \mid a_j \in (Q^+)^\mathcal{A}\}$ (in particular, $i \leftarrow 1$ if $(Q^+)^\mathcal{A} = \emptyset$)
while $i \leq |A|$ **do**
  if $\big((\mathcal{A}, (Q^+)^\mathcal{A} \cup \{a_i\}), \ell\big) \in p\text{-WD}_{(\psi \wedge \textit{constr})}$
    then return $\textsc{Constr}_\psi(\mathcal{A}, (Q^+)^\mathcal{A} \cup \{a_i\}, \ell)$
  $i \leftarrow i + 1$

$\textsc{List}_\psi(\mathcal{A}, (Q^+)^\mathcal{A}, (Q^-)^\mathcal{A}, \ell)$
// a τ-structure $\mathcal{A}$ with an ordering, $(Q^+)^\mathcal{A} \cup (Q^-)^\mathcal{A} \subseteq A$, $(Q^+)^\mathcal{A} \cap (Q^-)^\mathcal{A} = \emptyset$,
// and $\ell \in \mathbb{N}$

if $\big((\mathcal{A}, (Q^+)^\mathcal{A}, (Q^-)^\mathcal{A}), \ell\big) \notin p\text{-WD}_{(\psi \wedge \textit{list})}$ **then** return
if $|(Q^+)^\mathcal{A}| = \ell$ **then** output $(Q^+)^\mathcal{A}$
    return
$i \leftarrow 1 + \max\{j \mid a_j \in (Q^+)^\mathcal{A} \cup (Q^-)^\mathcal{A}\}$
$\textsc{List}_\psi(\mathcal{A}, (Q^+)^\mathcal{A} \cup \{a_i\}, (Q^-)^\mathcal{A}, \ell)$
$\textsc{List}_\psi(\mathcal{A}, (Q^+)^\mathcal{A}, (Q^-)^\mathcal{A} \cup \{a_i\}, \ell)$
return

□

Clearly, the decision problem is reducible to the construction problem and to the listing problem. For $\psi(Z) \in \Pi_t$ the formulas $(\psi \wedge \textit{constr})$ and $(\psi \wedge \textit{list})$ are Π_t-formulas. Hence, by the previous theorem, if $\psi(Z) \in \Pi_t$ and $p\text{-WD}_\psi$ is W$[t]$-complete under fpt-reductions, then the problems $p\text{-WD}_\psi$, p-CONSTR-WD$_\psi$, and p-LIST-WD$_\psi$ are "Turing fpt-equivalent."

We turn to maximality/minimilaty problems. As we will see in this section, it is not hard to show the following using the techniques developed so far:

> If by the previous results we know that p-MAXIMAL-WD$_\varphi$ is in W$[t]$, then the construction problem p-CONSTR-MAXIMAL-WD$_\varphi$ and the listing problem p-LIST-MAXIMAL-WD$_\varphi$ are solvable by polynomial time algorithms with an oracle to a decision problem in W$[t]$.

The same holds for minimality problems. For maximality problems the precise result reads as follows (compare it with Theorem 2):

Theorem 49. *Let $t \geq 1$ and $\varphi(Z)$ a Π_t-formula negative in Z.*

(a) If t is odd, then there are a polynomial time algorithm and a polynomial time delay algorithm, both with fpt-queries to an oracle for a problem in W$[t+1]$, *solving p-CONSTR-MAXIMAL-WD$_\varphi$ and p-LIST-MAXIMAL-WD$_\varphi$, respectively.*

(b) If t is even, then there are a polynomial time algorithm and a polynomial time delay algorithm, both with fpt-queries to an oracle for a problem in W$[t]$, *solving p-CONSTR-MAXIMAL-WD$_\varphi$ and p-LIST-MAXIMAL-WD$_\varphi$, respectively.*

Proof: Let $\varphi(Z)$ be a Π_t-formula negative in Z. We want to apply Theorem 48. So we only need a formula $\psi(Z)$ of the appropriate logical complexity expressing that "Z is a maximal solution of $\varphi(Z)$."

We first turn to (a). We can take as $\psi(Z)$ the Π_{t+1}-formula *1-max-*$\varphi(Z)$ (cf. Lemma 28). Theorem 48 shows that we can solve our problems with an oracle to p-WD$_{\textit{1-max-}\varphi\wedge\textit{constr}}$ or to p-WD$_{\textit{1-max-}\varphi\wedge\textit{list}}$, problems in W$[t+1]$ as *1-max-*$\varphi(Z) \wedge$ *constr*(Z) and *1-max-*$\varphi \wedge$ *list* are Π_{t+1}-formulas, too.

Now let t be even. Of course, again we could use an oracle to p-WD$_{\textit{1-max-}\varphi\wedge\textit{constr}}$ or to p-WD$_{\textit{1-max-}\varphi\wedge\textit{list}}$, but now we want an oracle to a problem in W$[t]$. We shall see that as such we can take p-MC$(\Sigma_{t,2})$. It suffices (by Lemma 15 and Lemma 16) to consider the construction and listing problems for maximal solutions of p-WSAT$(\Gamma^-_{t,d})$ instead of that for p-WD$_\varphi$. We look at the proof of Lemma 36 and use its notation. We see that in the structures considered there the formula

$$\exists y\Big(\big(\bigwedge_{i\in[k]} \textit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j\in[k]\\ i\neq j}} x_i \neq x_j \wedge \textit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \dots, x_k, x_k)\big) \wedge \forall x_{k+1}\big((\textit{VAR}\, x_{k+1} \wedge \bigwedge_{i\in[k]} x_i \neq x_{k+1}) \to \neg\psi_{t-2}(y, x_1, \dots, x_k, x_{k+1})\big)\Big).$$

expresses that $\{x_1, \dots, x_k\}$ is a maximal satisfying assignment. Now the proof of Theorem 48 can be adapted to yield our claims by using instead of, say, $\psi(Z) \wedge$ *list*(Z) the formula

$$\exists y\Big(\big(\bigwedge_{i\in[k]} \textit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j\in[k]\\ i\neq j}} x_i \neq x_j \wedge \textit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \dots, x_k, x_k)\big) \wedge \forall x_{k+1}\big((\textit{VAR}\, x_{k+1} \wedge \bigwedge_{i\in[k]} x_i \neq x_{k+1}) \to \neg\psi_{t-2}(y, x_1, \dots, x_k, x_{k+1})\big)\Big) \wedge \forall x\big((Q^+x \to \bigvee_{i\in[k]} x = x_i) \wedge (Q^-x \to \bigwedge_{i\in[k]} \neg x = x_i)\big),$$

which as in the proof of Lemma 36 can be replaced by a $\Sigma_{t,2}$-formula. Therefore, instead of an oracle to p-WD$_{\psi\wedge\textit{list}}$ we now need an oracle to p-MC$(\Sigma_{t,2})$. □

Remark 50. The preceding theorem deals with $\varphi(Z)$ negative in Z. Here we consider the case of an arbitrary $\varphi(Z)$. For such a formula $\varphi(Z)$ we set

$$\varphi^*(Z) := \varphi(Z \cup Q) \wedge \forall x(Zx \to \neg Qx),$$

where Q is a new unary relation symbol and $\varphi(Z \cup Q)$ is obtained from $\varphi(Z)$ by replacing every atomic formula of the form Zy by $(Zy \vee Qy)$. Then we get:

(a) There is a polynomial algorithm with fpt-queries to an oracle for p-MAXIMAL-WD$_{\varphi^*}$ solving the construction problem p-CONSTR-MAXIMAL-WD$_\varphi$.

(b) There is an fpt total time algorithm with an oracle to p-MAXIMAL-WD$_{\varphi^*}$ solving the listing problem p-LIST-MAXIMAL-WD$_\varphi$.

We present a proof for (b): Let $(\mathcal{A}, k)$ be an instance of p-MAXIMAL-WD$_\varphi$. We assume the universe A of $\mathcal{A}$ is ordered. We denote by a_i the ith element of A in this ordering. Then LIST-MAX$_\varphi$ on input $(\mathcal{A}, \emptyset, k)$ outputs all solutions in the induced lexicographical order:

```
LIST-MAX_φ(A, Q^A, ℓ)
// A an ordered τ-structure, Q^A a subset of A, ℓ ∈ ℕ.
if ((A, Q^A), ℓ) ∉ p-MAXIMAL-WDφ_max then return
if ℓ = 0 then output Q^A
        return
i ← max{j | a_j ∈ Q^A} + 1
while i ≤ |A| do
    LIST-MAX_φ(A, Q^A ∪ {a_i}, ℓ − 1)
return
```

Let $\{S_1, \ldots, S_m\}$ be the maximal solutions of $\varphi(Z)$ in $\mathcal{A}$ of size k for some $m \geq 0$. It is not hard to show that after line 1 of LIST-MAX$_\varphi$, $Q^{\mathcal{A}}$ has to be a subset of some S_i. Therefore, the total running time of LIST-MAX$_\varphi(\mathcal{A}, \emptyset, k)$ is bounded by

$$O(2^k \cdot m \cdot |A|).$$

Note that for $\varphi(Z)$ negative in Z the formula $\varphi^*(Z)$ is negative in Z, too, hence (a) yields part (a) of the preceding theorem. The result in (b) is slightly weaker, as we only get an fpt total time algorithm. By the way the proof method presented for part (b) of Theorem 49 generalizes to minimality problems (see Theorem 51), while we do not see any formula taking over the role of $\varphi^*(Z)$ for minimality problems.

We turn to minimality problems. Recall that the case $t = 1$ was already solved by Theorem 10.

Theorem 51. *Let $t \geq 2$ and $\varphi(Z)$ a Π_t-formula.*

(a) There are a polynomial time algorithm and a polynomial time delay algorithm, both with fpt-queries to an oracle for a problem in W$[t]$*, solving the construction problem* p-CONSTR-MINIMAL-WD$_\varphi$ *and the listing problem* p-LIST-MINIMAL-WD$_\varphi$*, respectively.*

(b) If t is odd, then there are a polynomial time algorithm and a polynomial time delay algorithm, both with fpt-queries to an oracle for a problem in W$[t-1]$*, solving* p-CONSTR-MINIMAL-WD$_\varphi$ *and* p-LIST-MINIMAL-WD$_\varphi$*, respectively.*

Since the proof is similar to that of part (b) of the preceding theorem (now using the proofs of Lemma 41 and Lemma 43 instead of Lemma 36) we omit it.

9. Counting problems

We have presented most proofs so far in such a way that they generalize (with small additions or corrections) to the counting context. But it arises the problem that some of the tools we have used have not been proven for the counting framework in the literature. In this section we start stating the results we get for counting maximal and (non-)minimal solutions, then we discuss some examples. Afterwards we will present the precise counting versions of the tools. Concerning the proofs we will be very sketchy and will only mention substantial changes or additional arguments that are necessary in Subsection 9.4.

We use the notations and results developed for counting problems in [18] (compare also [16]). For example, the counting version of p-WD$_\varphi$ is:

p-#WD$_\varphi$	
Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Compute the number of solutions of $\varphi(Z)$ in $\mathcal{A}$ of size k.

For counting problems $\mathcal{P}$ and $\mathcal{P}'$ we write $\mathcal{P} \leq^{\text{fpt}} \mathcal{P}'$ if there is an fpt *parsimonious* reduction from $\mathcal{P}$ to $\mathcal{P}'$.

9.1. Counting versions of the results.

For counting maximal, minimal and non-minimal solutions (we address the counting of non-maximal solutions in Subsection 9.5) we obtain the following analogues of the results for the decision problems, namely :

Theorem 52. *If $t \geq 1$ is odd, then*

$$\begin{aligned}\#\mathrm{W}[t+1] &= \left[\{p\text{-\#MAXIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula negative in } Z\}\right]^{\text{fpt}} \\ &= \left[\{p\text{-\#MAXIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_{t+1}\textit{-formula negative in } Z\}\right]^{\text{fpt}}.\end{aligned}$$

Theorem 53. *(a) If $t \geq 2$, then*

$$\#\mathrm{W}[t] = \left[\{p\text{-\#MINIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula}\}\right]^{\text{fpt}}.$$

(b) If $t \geq 2$ is even, then

$$\#\mathrm{W}[t] = \left[\{p\text{-\#MINIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula positive in } Z\}\right]^{\text{fpt}}$$
$$= \left[\{p\text{-\#MINIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_{t+1}\textit{-formula positive in } Z\}\right]^{\text{fpt}}.$$

(c) p-#MINIMAL-WD$_\varphi \in$ FPT *for every* Π_1*-formula* $\varphi(Z)$*.*

Theorem 54. *(a) If $t \geq 1$, then*

$$\#\mathrm{W}[t] = \left[\{p\text{-\#NON-MINIMAL-WD}_\varphi \mid \varphi(Z) \textit{ a } \Pi_t\textit{-formula}\}\right]^{\text{fpt}}.$$

(b) If $t \geq 2$ is even, then p-#NON-MINIMAL-WD$_\varphi \in \#\mathrm{W}[t]$ *for every* Π_{t+1}*-formula positive in* Z*.*

Note that part (b) of the preceding theorem is weaker than part (b) of Theorem 47. We discuss the problem whether the full analogue holds in Subsection 9.5.

9.2. Examples. First we show the $\#\mathrm{W}[2]$-completeness of two concrete counting problems:

Proposition 55. p-#MAXIMAL-INDEPENDENT-SET *and* p-#MAXIMAL-CLIQUE-OR-INDEPENDENT-SET *are* $\#\mathrm{W}[2]$*-complete under fpt parsimonious reductions.*

Proof: The fpt parsimonious reduction of [16] from the $\#\mathrm{W}[2]$-complete problem p-#$(\Pi^0_{1,1}[2])$ (we define this class of formulas below) to p-#DOMINATING-SET is a reduction to p-#INDEPENDENT-DOMINATING-SET, since the relevant dominating sets of the graph are independent sets, too. Thus p-#MAXIMAL-INDEPENDENT-SET ($=$ p-#INDEPENDENT-DOMINATING-SET; cf. the footnote on page 6) is $\#\mathrm{W}[2]$-hard under fpt parsimonious reductions. Furthermore,

$$p\text{-\#MAXIMAL-INDEPENDENT-SET} \leq^{\text{fpt}} p\text{-\#MAXIMAL-CLIQUE}$$
$$\leq^{\text{fpt}} p\text{-\#MAXIMAL-CLIQUE-OR-INDEPENDENT-SET},$$

since the corresponding reductions in Proposition 9 and in Theorem 37 also are fpt parsimonious reductions of the corresponding counting problems. But as the proof of Theorem 37 shows, there is a Π_1-formula $\psi(Z)$ negative in Z such that

$$p\text{-\#MAXIMAL-CLIQUE-OR-INDEPENDENT-SET} = p\text{-\#MAXIMAL-WD}_\psi.$$

Thus, p-#MAXIMAL-CLIQUE-OR-INDEPENDENT-SET $\in \#\mathrm{W}[2]$ by Theorem 52. □

In [1] it was shown that p-#CLIQUE-OR-INDEPENDENT-SET is #W[1]-hard under Turing reductions. Unless W[1] $=$ FPT it is not #W[1]-hard under fpt reductions, since the corresponding decision version is in FPT (cf. Theorem 37).

Modifying the reduction of p-DOMINATING-SET to p-MINIMAL-DOMINATING-SET presented in Proposition 11 appropriately, one gets:

Proposition 56. p-#MINIMAL-DOMINATING-SET *is* #W[2]*-complete under fpt parsimonious reductions.*

9.3. Counting versions of tools. We shall need two additional hierarchies $(\Sigma^0_{t,u})_{t\geq 1}$ and $(\Pi^0_{t,u})_{t\geq 1}$ for every fixed $u \geq 1$. We let $\Sigma^0_{0,u} = \Pi^0_{0,u}$ be the class of quantifier-free formulas. For $t \geq 1$, we let $\Pi^0_{t,u}$ be the class of all first-order formulas of the form $\forall x_1 \dots \forall x_k\ \psi$, where $k \leq u$ and $\psi \in \Sigma^0_{t-1,u}$. Similarly, $\Sigma^0_{t,u}$ is the class of all first-order formulas of the form $\exists x_1 \dots \exists x_k\ \psi$, where $k \leq u$ and $\psi \in \Pi^0_{t-1,u}$. Thus for $t \geq 1$, $\Sigma_{t,u}$ is the class of all first-order formulas of the form $\exists x_1 \dots \exists x_k\ \psi$, where $k \in \mathbb{N}$ and $\psi \in \Pi^0_{t-1,u}$.

For a class Φ of formulas and $s \in \mathbb{N}$ we denote by $\Phi[s]$ the class of formulas $\varphi \in \Phi$ in a vocabulary of arity $\leq s$. If φ is a first-order formula we write

$$\varphi\langle x_1, \dots, x_n\rangle \quad \text{and} \quad \varphi(x_1, \dots, x_n)$$

to indicate that the free variables in φ are $x_1, \dots, x_n$ and are among $x_1, \dots, x_n$, respectively.

If $\mathcal{A}$ is a τ-structure, and $\varphi\langle x_1, \dots, x_n\rangle$ is a formula of vocabulary τ, then we let

$$\varphi(\mathcal{A}) := \big\{(a_1, \dots, a_n) \in A^n \mid \mathcal{A} \models \varphi(a_1, \dots, a_n)\big\}.$$

Thus $|\varphi(\mathcal{A})|$ is the number of tuples satisfying φ. For a class Φ of formulas we denote by p-#$\mathrm{MC}(\Phi)$ the problem:

p-#$\mathrm{MC}(\Phi)$	
Input:	A structure $\mathcal{A}$ and $\varphi\langle x_1, \dots, x_n\rangle \in \Phi$.
Parameter:	$\lvert\varphi\rvert$.
Problem:	Compute $\lvert\varphi(\mathcal{A})\rvert$.

It is shown in [16]:

Proposition 57. *For $s \geq 2$ and $u \geq 1$, the problems p-#$\mathrm{MC}(\Pi^0_{0,u}[s])$ and p-#$\mathrm{MC}(\Pi^0_{1,u}[s])$ are $\#\mathrm{W}[1]$ and $\#\mathrm{W}[2]$-complete under fpt parsimonious reductions, respectively.*

In the proofs of the previous sections we often made use of Theorem 1 and Theorem 4. The following theorem contains the counting analogues:

Theorem 58. *Let $t \geq 1$.*

(a) $\#\mathrm{W}[t] = \big[\{p\text{-}\#\mathrm{WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula}\}\big]^{\mathrm{fpt}}$.

(b) If t is odd, then

$$\#\mathrm{W}[t] = \big[\{p\text{-}\#\mathrm{WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula negative in } Z\}\big]^{\mathrm{fpt}}$$
$$= \big[\{p\text{-}\#\mathrm{WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_{t+1}\text{-formula negative in } Z\}\big]^{\mathrm{fpt}}.$$

(c) If t is even, then

$$\#\mathrm{W}[t] = \big[\{p\text{-}\#\mathrm{WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_t\text{-formula positive in } Z\}\big]^{\mathrm{fpt}}$$
$$= \big[\{p\text{-}\#\mathrm{WD}_\varphi \mid \varphi(Z) \text{ a } \Pi_{t+1}\text{-formula positive in } Z\}\big]^{\mathrm{fpt}}.$$

(d) Let $s \geq 2$ and $t, u \geq 1$. Then p-#$\mathrm{MC}(\Pi^0_{t-1,u}[s])$ is #$\mathrm{W}[t]$-complete under fpt parsimonious reductions.

(e) Let $t, d \geq 1$ and $t + d \geq 3$. Then p-#$\mathrm{WSAT}(\Gamma_{t,d})$ is #$\mathrm{W}[t]$-complete under fpt parsimonious reductions.

(f) Let $t, d \geq 1$ and $t + d \geq 3$. If t is even (odd), then p-#$\mathrm{WSAT}(\Gamma^+_{t,d})$ (p-#$\mathrm{WSAT}(\Gamma^-_{t,d})$) is #$\mathrm{W}[t]$-complete under fpt parsimonious reductions.

It should not be difficult to derive the statements of this theorem for a reader familiar with the proofs of these results for the decision problems in [18].

We also need the counting version of Lemma 30:

Lemma 59. *Let $t \geq 2$. Then:*

(a) p-#$\mathrm{PSAT}(\Gamma_{t,1})$ is #$\mathrm{W}[t]$-complete under fpt parsimonious reductions.

(b) For even t, the problem p-#$\mathrm{PSAT}(\Gamma^+_{t,1})$ is #$\mathrm{W}[t]$-complete under fpt parsimonious reductions.

Proof: The reductions in Lemma 7 and Lemma 8 of [19] are parsimonious. □

It has been pointed out in [27] that FPT $\not\subseteq$ $\#\mathrm{W}[1]$ for the "liberal" definition of the class FPT of counting problems given in [16, 18]. In fact, by this definition a parameterized counting problem (F, κ) is in FPT if there is an fpt-algorithm with respect to κ that computes F. Such functions can have "quite big values." On the other hand

$$\#\mathrm{W}[t] := \big[\{p\text{-}\#\mathrm{WD}_\varphi \mid \varphi \in \Pi_t\}\big]^{\mathrm{fpt}}$$

and the number of solutions of an instance $(\mathcal{A}, \varphi)$ of p-#WD_φ is bounded by $|A|^k$. However, every parameterized counting problem in FPT relevant in the context of this paper is in $\#\mathrm{W}[1]$, as we show:

Proposition 60. *Let* (F, κ) *with* $F : \Sigma^* \to \mathbb{N}$ *be a parameterized counting problem* (F, κ) *in* FPT. *Then the following are equivalent.*

(i) $(F, \kappa) \in \#\mathrm{W}[1]$.

(ii) There is a computable function g *such that for all* $x \in \Sigma^*$ *distinct from the empty word*

$$F(x) < (2|x|)^{g(\kappa(x))}.$$

Proof: We sketch a proof for the reader familiar with the exposition in [18]. There it is shown that an arbitrary parameterized counting problem (F, κ) is in $\#\mathrm{W}[1]$ if and only if there is a tail-nondeterministic κ-restricted program $\mathbb{P}$ for an NRAM (a nondeterministic random access machine) such that for every $x \in \Sigma^*$, the value $F(x)$ is the number of accepting runs of $\mathbb{P}$ on input x. This immediately yields the implication from (i) to (ii).

For the reverse implication assume that $(F, \kappa) \in$ FPT. Let $x \in \Sigma^*$, $n := |x|$ and $k := g(\kappa(x))$. By assumption, $F(x) < (2n)^k$, hence the $(2n)$-ary representation of $F(x)$ has length $\leq k$. An NRAM program $\mathbb{P}$ witnessing that $F \in \#\mathrm{W}[1]$ first computes $F(x)$ and stores the corresponding $(2n)$-ary representation. Then $\mathbb{P}$ guesses a number $i \leq k$. If the ith "bit" of $F(x)$ is 0, it rejects. Otherwise, $\mathbb{P}$ enters a loop that is called i times and then $\mathbb{P}$ stops accepting; each run of the loop consists of a single step, namely of the guess of a number in $[2n]$. □

9.4. Proofs of the counting versions. Similarly as for the decision problems the correspondence between weighted satisfiability problems and Fagin-definable problems plays a very important role in the counting context. To prove a counting version of Corollary 17 we need to strengthen Lemma 15 to the following.

Lemma 61. *Let* $t, d \geq 1$ *and* $\varphi(Z)$ *be a* $\Pi_{t/d}$*-formula of vocabulary* τ. *Then there is a polynomial time algorithm associating with every* τ*-structure* $\mathcal{A}$ *a propositional formula* $\beta \in \Gamma_{t,\max(d,2)}$ *such that* $\mathrm{Var}(\beta) = \{X_a \mid a \in A\}$ *and for all* $S \subseteq A$

$$\mathcal{A} \models \varphi(S) \iff \{X_b \mid b \in S\} \text{ satisfies } \beta.$$

Furthermore

(a) If $\varphi(Z)$ *is positive in* Z*, then the polynomial time algorith can be chosen such that* $\beta \in \Gamma^+_{t,\max(d,2)}$ *with* $\mathrm{Var}(\beta) = \{X_a \mid a \in A\} \cup \{Y\}$ *(where* Y *is new propositional variable) and such that*
  - *for all* $S \subseteq A$ $(\mathcal{A} \models \varphi(S) \iff \{X_b \mid b \in S\} \cup \{Y\}$ *satisfies* $\beta)$*;*
  - *every assignment satisfying* β *and distinct from* $\{X_a \mid a \in A\}$ *must set* Y *to* TRUE.

(b) If $\varphi(Z)$ *is negative in* Z*, then the polynomial time algorithm can be chosen such that* $\beta \in \Gamma^-_{t,\max(d,2)}$ *with* $\mathrm{Var}(\beta) = \{X_a \mid a \in A\} \cup \{Y\}$ *and such that*
  - *for all* $S \subseteq A$ $(\mathcal{A} \models \varphi(S) \iff \{X_b \mid b \in S\}$ *satisfies* $\beta)$*;*
  - *every assignment satisfying* β *and distinct from* $\{Y\}$ *must set* Y *to* FALSE.

Proof: We show how to get the desired formula γ from the formula α of Lemma 15. Recall that $\mathrm{Var}(\alpha) \subseteq \{X_a \mid a \in A\}$. We let

$$\beta := \alpha \wedge \bigwedge_{a \in A} (X_a \vee \neg X_a).$$

In case α is positive, we take

$$\beta := \alpha \wedge \bigwedge_{a \in A} (X_a \vee Y);$$

and for negative α

$$\beta := \alpha \wedge \bigwedge_{a \in A} (\neg X_a \vee \neg Y).$$ □

In the proof of the translation of Lemma 41 to the counting context (Lemma 62 below) we more or less have to use all the tricks which are necessary in the corresponding task for other results. So we present a proof of it. Moreover an additional argument is necessary in order to obtain some of the extensions; for example, Lemma 63 is needed to get Theorem 54 (b).

Lemma 62. *Let $t > 1$ be odd and $d \geq 1$. Then p-#MINIMAL-WSAT$(\Gamma^+_{t,d}) \in$ #W$[t-1]$.*

Proof: We show that p-#MINIMAL-WSAT$(\Gamma^+_{t,d}) \leq^{\text{fpt}} p$-#MC$(\Pi^0_{t-2,4}[d+2])$, which yields the claim by Theorem 58 (d). We use Lemma 35 and its terminology. Let (α, k) be an instance of p-#MINIMAL-WSAT$(\Gamma^+_{t,d})$ and let $\{X_1, \ldots, X_n\}$ be the set of variables of α. Let the Π_{t-2}-formula ψ_{t-2} and the structure $\mathcal{A}$ be as in Lemma 35. Then $[n] \subseteq A$. We add the natural ordering on $[n]$ to $\mathcal{A}$, thereby obtaining a structure $\mathcal{B}$. Then, by Lemma 35, we have

– For $a \in ROOT^{\mathcal{B}}$ and arbitrary $m_1, \ldots, m_k \in [n]$ with $m_1 < \ldots < m_k$

$$\{X_{m_1}, \ldots, X_{m_k}\} \text{ satisfies } \alpha \quad \Longleftrightarrow \quad \mathcal{B} \models \psi_{t-2}(a, m_1, \ldots, m_k). \tag{34}$$

and hence $\{X_{m_1}, \ldots, X_{m_k}\}$ is a minimal satisfying assignment of α if and only if

$$\mathcal{B} \models \Big(\big(\bigwedge_{i \in [k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j \in [k] \\ i<j}} x_i < x_j \wedge \mathit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \ldots, x_k) \big) \wedge \bigwedge_{i \in [k]} \neg \psi_{t-2}(y, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, x_{i+1}) \big) \Big)(a, m_1, \ldots, m_k),$$

where $x_{k+1} = x_1$. Let $\psi_{t-2}(y, x_1, \ldots, x_k) = \forall u \psi'(u, y, x_1, \ldots, x_k)$. Let $\bar{x} = x_1 \ldots x_k$, $\bar{u} = u_1 \ldots u_k$, and for $i \in [k]$ set

$$\bar{v}_i := x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, x_{i+1}.$$

We introduce the formula:

$$\rho(y, \bar{x}, \bar{u}) := \Big(\big(\bigwedge_{i \in [k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j \in [k] \\ i \neq j}} x_i < x_j \wedge \mathit{ROOT}\, y \wedge \psi_{t-2}(y, x_1, \ldots, x_k) \big) \wedge \bigwedge_{i \in [k]} \Big(\neg \psi'(u_i, y, \bar{v}_i) \wedge \forall z \big(z < u_i \rightarrow \psi'(z, y, \bar{v}_i) \big) \Big) \Big).$$

Hence, $\{X_{m_1}, \ldots, X_{m_k}\}$ with $m_1 < \ldots < m_k$ is a minimal satisfying assignment of α if and only if there is a tuple $\bar{b} \in B^k$ such that $\mathcal{B} \models \rho(a, m_1, \ldots, m_k, \bar{b})$; moreover, in the positive case, this tuple is uniquely determined. Thus the number of minimal satisfying assignments of α coincides with the number of tuples satisfying $\rho(y, \bar{x}, \bar{u})$. So it suffices to show that $\rho(y, \bar{x}, \bar{u})$ is equivalent to a $\Pi^0_{t-2,4}$-formula. First note that $\Big(\neg \psi'(u_i, \bar{v}_i) \wedge \forall z \big(z \leq u_i \rightarrow \psi'(z, \bar{v}_i) \big) \Big)$ is equivalent to a $\Pi^0_{t-2,2}$-formula. Applying Lemma 40 repeatedly as it was done for the corresponding formula in the proof of Lemma 41, one easily sees that ρ is equivalent to a $\Pi^0_{t-2,4}$-formula. □

Lemma 63. *Let $t \geq 3$ be odd. Then p-#NON-MINIMAL-WSAT$(\Gamma^+_{t,1}) \in$ #W$[t-1]$.*

Proof: We show our claim by proving p-#NON-MINIMAL-WSAT$(\Gamma^+_{t,d}) \leq^{\text{fpt}} p$-#MC$(\Pi^0_{t-2,4}[d+2])$. Let (α, k) be an instance of p-#NON-MINIMAL-WSAT$(\Gamma^+_{t,d})$ and let $\{X_1, \ldots, X_n\}$ be the set of variables of α. Let the Π_{t-2}-formula $\psi_{t-2}(y, x_1, \ldots, x_k)$ and the structure $\mathcal{A}$ be as in the preceding proof. Then $[n] \subseteq A$ and as there we add the natural ordering on $[n]$ to $\mathcal{A}$, thereby obtaining a structure $\mathcal{B}$. Then (34) holds. For $V \subset \{x_1, \ldots, x_k\}$, say $V = \{x_{i_1}, \ldots, x_{i_\ell}\}$ with $i_1 < \ldots < i_\ell$, we set

$$\psi_{t-2}(y, \langle V \rangle) := \psi_{t-2}(y, x_{i_1}, \ldots, x_{i_\ell}, x_{i_\ell}, \ldots, x_{i_\ell}).$$

Then, by (34), for $a \in ROOT^{\mathcal{B}}$ and arbitrary $m_1, \ldots, m_k \in [n]$ with $m_1 < \ldots < m_k$, the assignment $\{X_{m_1}, \ldots, X_{m_k}\}$ is a non-minimal satisfying assignment of α if and only if

$$\mathcal{B} \models \Big(\bigwedge_{i \in [k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j \in [k] \\ i<j}} x_i < x_j \wedge \mathit{ROOT}\, y \wedge \bigvee_{V \subset \{x_1, \ldots, x_k\}} \psi_{t-2}(y, \langle V \rangle) \Big)(a, m_1, \ldots, m_k).$$

Note that a non-minimal satisfying assignment maybe an extension of various minimal satisfying assignments. It is convenient to “fix the first one” by passing to the formula

$$\bigwedge_{i\in[k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j\in[k]\\ i<j}} x_i < x_j \wedge \mathit{ROOT}\, y \wedge \bigvee_{V\subset\{x_1,\dots,x_k\}} \Big(\psi_{t-2}(y,\langle V\rangle) \wedge \bigwedge_{V'<V} \neg\psi_{t-2}(y,\langle V'\rangle)\Big)$$

(here $V' < V$ means that V' is less than V in the lexicographic order induced by the order of the variables). Again let $\psi_{t-2}(y, x_1, \dots, x_k) = \forall u\psi'(u, y, x_1, \dots, x_k)$. Take new variables z and u_V for $V \subset \{x_1, \dots, x_k\}$ and consider the formula

$$\bigwedge_{i\in[k]} \mathit{VAR}\, x_i \wedge \bigwedge_{\substack{i,j\in[k]\\ i<j}} x_i < x_j \wedge \mathit{ROOT}\, y \wedge$$

$$\bigvee_{V\subset\{x_1,\dots,x_k\}} \Big(R_V z \wedge \psi_{t-2}(y,\langle V\rangle) \wedge \bigwedge_{V\le V'} R_V u_{V'} \wedge \bigwedge_{V'<V} \neg\psi'_{t-2}(u_{V'}, y, \langle V'\rangle)\Big).$$

Now the proof is finished as the preceding one. □

9.5. Differences between counting and decision problems. Why cannot we generalize the result for the non-maximality problems contained in Theorem 38 to the counting context and why is part (b) of Theorem 54 weaker than the corresponding statement in Theorem 47? To derive the (non-counting) results we used the equivalences (28) and (33); both do not survive for the counting problems, that is, the number of solutions for the problems on the left side of the equivalences might be different from the number of solutions for the problems on the right side.

We already mentioned in the Introduction that if p-#NON-MAXIMAL-WD$_\varphi \in$ #W$[t]$ for all $\varphi(Z) \in \Pi_t$ negative in Z, then for odd t some #W$[t+1]$-complete problem would be solvable by an fpt-algorithm with an oracle to some problem in #W$[t]$. We can show even more. The following theorem contains the corresponding results for non-maximality and non-minimality problems. By $[\mathrm{FPT}]^{\mathrm{W[P]\text{-}rfpt}}$ we denote the closure of FPT under quite weak randomized reductions (a precise definition of this class is given later).

Theorem 64. *(a) Let $t \ge 1$ and $\varphi(Z)$ a Π_t-formula negative in Z. Then*

- *(i)* p-#NON-MAXIMAL-WD$_\varphi \in$ #W$[t+1]$.
- *(ii) If* W$[t] \ne$ W$[t+1]$*, then* p-#NON-MAXIMAL-WD$_\varphi$ *is not* #W$[t+1]$*-hard under fpt parsimonious reductions.*
- *(iii) If* W$[1] \not\subseteq [\mathrm{FPT}]^{\mathrm{W[P]\text{-}rfpt}}$*, then* p-#NON-MAXIMAL-WD$_\varphi$ *is not* #W$[1]$*-hard under fpt parsimonious reductions.*

(b) Let $t \ge 1$ and $\varphi(Z)$ a Π_t-formula positive in Z. If W$[1] \not\subseteq [\mathrm{FPT}]^{\mathrm{W[P]\text{-}rfpt}}$*, then* p-#NON-MINIMAL-WD$_\varphi$ *is not* #W$[1]$*-hard under fpt parsimonious reductions.*

In the proof of part (a)(1) we will use the next lemma. While p-WD$_\varphi \in$ W$[t-1]$ for every Σ_t-formula $\varphi(Z)$ (cf. Lemma 5.4 in [18]), only a weaker statement is true for the counting problems:

Lemma 65. *Let $t \ge 1$ and $\varphi(Z)$ a Σ_t-formula. Then* p-#WD$_\varphi \in$ #W$[t]$.

Proof: Let $\varphi(Z) \in \Sigma_t$ be of vocabulary τ, say,

$$\varphi(Z) = \exists x_1 \dots \exists x_m\ \psi(x_1, \dots, x_m, Z),$$

where $\psi \in \Pi_{t-1}$. We assume that the variables $x_1, \dots, x_m$ are not quantified in ψ. Let $z_1, \dots, z_m$ be new variables and define ψ_1 to be the formula obtained from ψ by substituting every occurrence of x_i for $i \in [m]$ by z_i, Moreover, let Y be a new $(m+1)$-ary relation symbol. Let ψ' and ψ'_1 be the formulas obtained from ψ and ψ_1 by replacing each subformula of the form Zy by $Yx_1 \dots x_m y$ and $Yz_1 \dots z_m y$, respectively.

Let $\mathcal{A}$ be a τ-structure. We expand $\mathcal{A}$ with a total order $<$ on its universe A. Let

$$\chi := \forall x_1 \dots \forall x_{m+1} \forall y_1 \dots \forall y_{m+1} \big((Y x_1 \dots x_{m+1} \wedge Y y_1 \dots y_{m+1}) \to \bigwedge_{i \in [m]} x_i = y_i \big)$$

and let $\rho(Y)$ be the formula

$$\chi \wedge \forall x_1 \dots \forall x_{m+1} \big((Y x_1 \dots x_{m+1} \to \psi') \wedge \forall z_1 \dots \forall z_m (\textit{less}(z_1, \dots, z_m, x_1, \dots x_m) \to \neg \psi'_1) \big),$$

where

$$\textit{less}(\bar{z}, \bar{x}) := \bigvee_{i \in [m]} \big(z_i < x_i \wedge \bigwedge_{j \in [i-1]} z_j = x_j \big).$$

It is routine to check that for $k \geq 1$

$$\big| \{ S \mid S \subseteq A, |S| = k, \text{ and } \mathcal{A} \models \varphi(S) \} \big| = \big| \{ T \mid T \subseteq A^{m+1}, |T| = k, \text{ and } (\mathcal{A}, <) \models \rho(T) \} \big|.$$

Since ρ is equivalent to a Π_t-formula, this shows that p-#WD$_\varphi \in$ #W$[t]$. □

Proof of parts (a)(i) and (a)(ii) of Theorem 64: (a)(i) By antimonotonicity, p-NON-MAXIMAL-WD$_\varphi$ = p-WD$_\psi$, where

$$\psi(Z) := \varphi(Z) \wedge \exists x (\neg Z x \wedge \varphi(Z \cup \{x\})).$$

Since ψ is equivalent to a Σ_{t+1}-formula, we get p-#NON-MAXIMAL-WD$_\varphi \in$ #W$[t+1]$ by the previous lemma.

(a)(ii) Assume that p-#NON-MAXIMAL-WD$_\varphi$ is #W$[t+1]$-hard under fpt parsimonious reductions. Every fpt parsimonious reduction of the #W$[t+1]$-complete problem p-#WSAT$(\Gamma_{t+1,2})$ to p-#NON-MAXIMAL-WD$_\varphi$ yields an fpt-reduction of the W$[t+1]$-complete problem p-WSAT$(\Gamma_{t+1,2})$ to p-NON-MAXIMAL-WD$_\varphi$. However from Theorem 38 we know that the latter problem is in W$[t]$. Hence W$[t]$ = W$[t+1]$. □

We turn to the remaining parts of Theorem 64. We will need:

Lemma 66. *Let $(\mathcal{A}, k)$ be an instance of p-WD$_\varphi$.*

(a) If $\varphi(Z)$ is negative in Z and there is a unique non-maximal solution of $\varphi(Z)$ in $\mathcal{A}$ of size k, then $k = 0$.

(b) If $\varphi(Z)$ is positive in Z and there is a unique non-minimal solution of $\varphi(Z)$ in $\mathcal{A}$ of size k, then $k = |A|$.

Proof: Assume $\varphi(Z)$ is negative in Z and S is the unique non-maximal solution of $\varphi(Z)$ in $\mathcal{A}$ of size k. Choose a solution $T \supset S$ witnessing the non-maximality. By antimonotonicity, every subset of T of size k is a solution, too. By uniqueness of S, we get $S = \emptyset$ and hence, $k = 0$. The proof of (b) is similar. □

We need the following lemma. It is a consequence of a much more general result due to Müller [26], which generalizes the Valiant-Vazirani Lemma to the context of parameterized complexity.

Lemma 67. *There is a* W[P]*-randomized fpt-reduction R_{uni} from p-INDEPENDENT-SET to the problem p-UNIQUE-INDEPENDENTSET.*

Here

p-UNIQUE-INDEPENDENT-SET	
Input:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Does $\mathcal{G}$ have a unique independent set of size k?

We do not introduce the notion of W[P]-randomized fpt-reduction, but we introduce the class $[\text{FPT}]^{\text{W[P]-rfpt}}$, the class of problems solvable by a W[P]-randomized fpt-algorithm, thereby using some concepts defined in [18]:

Definition 68. We let $[\text{FPT}]^{\text{W[P]-rfpt}}$ be the class of all parameterized problems (P, κ) such that there is an exact κ-restricted tail-nondeterministic program $\mathbb{P}_0$ for an NRAM (a nondeterministic random access machine) having uniform guess bounds and there are a computable function f and a polynomial $p \in \mathbb{N}[X]$ such that for all instances x of (P, κ) we have:

– If $x \in P$, then $\Pr_x(\mathbb{P}_0 \text{ accepts } x) \geq \frac{1}{f(\kappa(x)) \cdot p(|x|)}$.

– If $x \notin P$, then $\Pr_x(\mathbb{P}_0 \text{ accepts } x) = 0$.

Thereby a program for an NRAM is *exact* if for every instance x it performs the same number of nondeterministic steps on every run on input x and it *has uniform guess bounds* if for every run on input x the contents of the register 0 (containing the bound for a nondeterministic step) is the same for any two nondeterministic steps. Moreover $\Pr_x$ denotes the uniform measure on the set of runs of the program $\mathbb{P}_0$ on input x.

How strong is the assumption $\text{W}[1] \not\subseteq [\text{FPT}]^{\text{W[P]-rfpt}}$ in Theorem 64 (a)(iii)? In the following discussion we assume that the reader is familiar with the classical complexity classes RP and BPP.

Proposition 69. *(a)* $[\text{FPT}]^{\text{W[P]-rfpt}} \subseteq$ para-RP.

(b) If PTIME $=$ RP, *then* FPT $= [\text{FPT}]^{\text{W[P]-rfpt}}$.

Proof: (a) One easily verifies that a parameterized problem (P, κ) is in para-RP if and only if there is a nondeterministic fpt time-bounded Turing machine $\mathbb{M}$ such that for each instance x:

– If $x \in P$, then $\Pr_x(\mathbb{M} \text{ accepts } x) \geq 1/2$.

– If $x \notin P$, then $\Pr_x(\mathbb{M} \text{ accepts } x) = 0$.

Now assume that an NRAM $\mathbb{P}_0$ together with a computable function f and a polynomial p witness that the parameterized problem (P, κ) is in $[\text{FPT}]^{\text{W[P]-rfpt}}$. Let $\mathbb{M}$ be a nondeterministic Turing machine that on every instance x simulates $f(\kappa(x)) \cdot p(|x|)$ many times a computation of $\mathbb{P}_0$ on input x and accepts if at least one of the computations was accepting. It is easy to see that $\mathbb{M}$ witnesses that $(P, \kappa) \in$ para-RP.

Since para-PTIME $=$ FPT, part (b) is immediate by (a). $\square$

By [23], one would conjecture that PTIME $=$ BPP and hence, PTIME $=$ RP. So, by the proposition, one conjectures that FPT $= [\text{FPT}]^{\text{W[P]-rfpt}}$ and hence, $\text{W}[1] \not\subseteq [\text{FPT}]^{\text{W[P]-rfpt}}$.

Proof of parts (a)(iii) and (b) of Theorem 64: For (a)(iii) let $t \geq 1$ and $\varphi(Z)$ be negative in Z. By contradiction assume that there is an fpt parsimonious reduction R_0 from the #W[1]-complete problem p-#INDEPENDENT-SET to p-#NON-MAXIMAL-WD$_\varphi$. Let R_{uni} be the reduction from p-INDEPENDENT-SET to p-UNIQUE-INDEPENDENTSET of Lemma 67. By Lemma 66(a) it is easy to see that for any instance $(\mathcal{G}, k)$ the following W[1]-randomized fpt algorithm decides whether $(\mathcal{G}, k) \in p$-INDEPENDENT-SET.

$(\mathcal{G}', k') \leftarrow R_{\text{uni}}(\mathcal{G}, k)$
$(\mathcal{A}, k_{\mathcal{A}}) \leftarrow R_0(\mathcal{G}', k')$
if $k_{\mathcal{A}} \neq 0$ **then** reject
for all $a \in A$ **do**
    if $\mathcal{A} \models \varphi(\{a\})$ **then** accept
Reject

Hence $\mathrm{W}[1] \subseteq [\mathrm{FPT}]^{\text{W[P]-rfpt}}$.

The proof of part (b) using Lemma 66 (b) is similar. □

In particular the preceding theorem shows that p-#NON-MAXIMAL-INDEPENDENT-SET is not $\#\mathrm{W}[1]$-hard under fpt parsimonious reductions (unless $\mathrm{W}[1] \subseteq [\mathrm{FPT}]^{\text{W[P]-rfpt}}$). On the other hand:

Proposition 70. *p-#NON-MAXIMAL-INDEPENDENT-SET is $\#\mathrm{W}[2]$-hard under fpt Turing reductions.*

Proof: We know that p-#MAXIMAL-INDEPENDENT-SET is $\#\mathrm{W}[2]$-hard under fpt parsimonious reductions (compare Proposition 55). Therefore by the equality (1) of the Introduction, it suffices to give an fpt Turing reduction from p-#INDEPENDENT-SET to p-#NON-MAXIMAL-INDEPENDENT-SET.

Let $\mathcal{G} = (V, E)$ be a graph and $k \in \mathbb{N}$. We define the graph $\mathcal{G}' = (V', E')$ by

$$\begin{aligned} V' :=& V \times [2] \times [k] \\ E' :=& \bigcup_{i \in [2], j \in [k]} \big\{\{(u,i,j),(v,i,j)\} \mid u, v \in V \text{ with } u \neq v\big\} \\ & \cup \bigcup_{j \in [k]} \big\{\{(u,1,j),(v,2,j) \mid u, v \in V \text{ with } u \neq v\big\} \\ & \cup \bigcup_{\substack{i_1, i_2 \in [2] \\ j_1, j_2 \in [k], j_1 < j_2}} \big\{\{(u,i_1,j_1),(v,i_2,j_2)\} \mid u = v \text{ or } \{u,v\} \in E\big\}. \end{aligned}$$

One easily verifies that:

– For any independent set $S = \{v_1, \dots v_k\}$ of $\mathcal{G}$ of size k, there are $k!$ distinct independent sets of size $2k$ of $\mathcal{G}'$ of the form
$$\big\{(u_1,1,1),(u_1,2,1),\dots,(u_k,1,k),(u_k,2,k)\big\} \tag{35}$$
with $\{v_1, \dots v_k\} = \{u_1, \dots u_k\}$;

– Any independent set of size $2k$ of $\mathcal{G}'$ is of the form (35) and thereby $\{u_1, \dots u_k\}$ is an independent set of $\mathcal{G}'$ of size k.

Clearly every independent set of $\mathcal{G}'$ of the form (35) contains $2k$ independent sets of size $2k-1$. On the other hand it is easy to verify that any independent set in $\mathcal{G}'$ of size $2k-1$ is contained in a unique independent set of size $2k$, which has to be of the form (35). Hence

$$\begin{aligned} &\text{the number of independent sets of size } k \text{ of } \mathcal{G} \\ &= \frac{1}{2k \cdot k!} \cdot \text{the number of non-maximal independent sets of size } (2k-1) \text{ of } \mathcal{G}'. \end{aligned}$$

This yields the desired reduction. □

10. Conclusions, extensions, and open problems

In this paper we have analyzed the relationship between the complexity of a parameterized problem and the corresponding maximality (minimality) problem asking for a solution of size k maximal (minimal) with respect to set inclusion. We believe that our results show that the notion Fagin-definability yields a very natural, far-reaching, and appropriate framework to define and study maximality and minimality problems. In the Introduction we have summarized our results for the maximal (minimal) weighted satisfiability problems. For many of them we do not see a direct proof using propositional formulas or circuits and the methods developed for them in parameterized complexity. In other words, the characterizations and the study of the classes of the W-hierarchy in terms of Fagin-definability and of model-checking problems turned out to be a very powerful tool in our context (at least for us).

We have seen that in terms of the W-hierarchy, some maximality problems increase the complexity while all minimality problems do not. We could extend nearly all results to the corresponding construction, listing, and counting problems; there were a few exceptions for counting problems.

The class W[P] and the classes of the A-hierarchy also have characterizations in terms of Fagin-definability and one can extend some of the results to them. For example, the class W[P], which often is seen as the parameterized analogue of NP, consists of the problems fpt-reducible to a problem of the form p-WD_φ, where $\varphi(Z)$ is a formula of fixpoint logic. Clearly, for such a formula φ the formula *1-max-*$\varphi(Z)$ in (21) is again a formula of fixpoint logic. Hence, from Lemma 28 we get:

– If $\varphi(Z)$ is a formula of fixpoint logic negative in Z, then p-MAXIMAL-$\mathrm{WD}_\varphi \in$ W[P].

But besides these "logical techniques" also the machine characterization of the class W[P] given in [6] is useful for the type of problems we are interested in. In particular, one can use it to derive the previous result and the first one of the following theorem, which contains some results for W[P] whose proofs we leave to the reader.

Theorem 71. *(a) If $\varphi(Z)$ is a formula of fixpoint logic, then* p-MINIMAL-$\mathrm{WD}_\varphi \in$ W[P].

(b) The maximality problems p-MAXIMAL-WSAT(CIRC) *and* p-MAXIMAL-WSAT(CIRC^-) *and the minimality problems* p-MINIMAL-WSAT(CIRC) *and* p-MINIMAL-WSAT(CIRC^+) *are* W[P]*-complete under fpt-reductions.*

Here CIRC^+ and CIRC^- denote the class of positive and of negative (cf. Subsection 2.3) circuits, respectively.

Let us close by mentioning two open problems. For odd $t > 1$ we could not settle the complexity of p-MAXIMAL-WSAT($\Gamma^-_{t,1}$). We know that it is a W[t]-hard problem contained in W[$t+1$].

The second problem concerns the EW-hierarchy introduced in [19]. Is p-MINIMAL-DOMINATING-SET EW[2]-complete? It is contained in EW[2,2] and, maybe, it is complete for this class.

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