Algorithms (XVIII)

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Chapter 10. Quantum algorithms

Qubits, superpositions, and measurement

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- either be in the ground state (the lowest energy configuration),
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We can use these two states to encode for bit values 0 and 1, respectively.

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Such a superposition is the basic unit of encoded information in quantum computers, called a qubit.

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The measurement would disturb the system, and the new state would actually become $|j\rangle$ or the jth excited state.

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$$|\alpha\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|1\rangle,$$

where $\sum_{x\in\{0,1\}^2} |\alpha_x|^2 = 1$.

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$$\text{Prob}\{\text{1st bit}=0\} = \text{Prob}\{00\} + \text{Prob}\{01\} = |\alpha_{00}|^2 + |\alpha_{01}|^2.$$

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The new state would be

$$|\alpha_{\text{new}}\rangle = \frac{\alpha_{00}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}|00\rangle + \frac{\alpha_{01}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}|01\rangle.$$

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Measuring the system only reveals n bits of information. The probability that the outcome is a particular n-bit string x is $|\alpha_x|^2$. And the new state after measurement is just $|x\rangle$.

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It is while the quantum system is not being watched that the quantum effects take over and we have the benefit of Nature working exponentially hard on our behalf.

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Observe that this output is random. As long as y corresponds to the right answer with high enough probability, we can repeat the whole process a few times to make the chance of failure miniscule.

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- Finding such a root is reduced to computing the order of a random integer modulo N.
- The order of an integer is precisely the period of a particular periodic superposition.
- Finally, periods of superpositions can be found by the quantum FFT.

The quantum Fourier transform

$$\begin{bmatrix} A(x_0) \\ A(x_1) \\ \vdots \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ & & \vdots & & \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$