Expander Graphs and Their Applications (I)

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Two Motivating Examples

Graph Reachability Problem

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  A BIG problem, since we can only read the external memory, not *write* it.

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A *space-efficient* algorithm:

```
DISTANCE_d(G, u, v, ℓ)
// G = (V, E) a graph, u, v ∈ V, and ℓ ∈ ℕ
  if u = v then return '1.'
  if ℓ = 0 then return '0.'
  for i = 1 to d do
        w ← the i-th neighbor of u.
        if DISTANCE_d(G, w, v, ℓ − 1) returns '1' then return
    '1.'
  return '0.'
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Let $G = (V, E)$ be a graph such that every vertex has degree at most d. Then for every $u, v \in V$ and $\ell \in \mathbb{N}$.

$$\textsc{Distance}_d(G, u, v, \ell) = 1 \iff \mathsf{dist}(u, v) \leq \ell.$$

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Let $c, d \in \mathbb{N}$. For (c, d)-good graphs $G = (V, E)$, the reachability problem can be solved in space

$$O(c \cdot \log |V| \cdot \log d) = O(\log |V|),$$

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i.e., in LogSpace.

Graph Reachability in LogSpace

Theorem (**Omer Reingold**, 2005)

There is an algorithm that solves the reachability problem on any graph $G = (V, E)$ *using space* $O(\log |V|)$.

Idea: *Reduce* the general problem to "good graphs,"

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There is an algorithm that solves the reachability problem on any graph $G = (V, E)$ using space $O(\log |V|)$.

Idea: *Reduce* the general problem to "good graphs," even stronger: Expander Graphs.

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Recall:

Max-IndependentSet	
Input:	A graph $G = (V, E)$.
Solution:	An independent set $I \subseteq V$.
Cost:	$\lvert I \rvert$.
Goal:	max.

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Theorem

There is a polynomial time 2-approximation for MIN-VERTEXCOVER.

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The major tool of Dinur's proof is Expander Graphs.

Introduction to Expander Graphs

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For every $S, T \subseteq V$ we let

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Then $e(S, T) := |E(S, T)|$. Note every edge can be *counted twice* except for those selfloops. Therefore it might not hold that $e(V, V) = 2|E|$, but

$$e(V, V) = d|V|$$

in case G is d-regular.

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- If G is a full binary tree, then $h(G) = 2/(|V| - 1)$.

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All the previous examples are not family of expander graphs.

Graph spectrum and an algebraic definition of expansion

Graphs as Matrices

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If G is d-regular and $A = A(G) = (a_{i,j})_{i,j \in [n]}$, then for every $i \in [n]$

$$\sum_{k \in [n]} a_{i,k} = d = \sum_{k \in [n]} a_{k,i}.$$

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for some $\lambda \in \mathbb{R}$. Then λ is an eigenvalue of A with corresponding eigenvector x.

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  Note, $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$.

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- $\lambda_1 = d$ and the corresponding eigenvectors is

$$v_1 = \frac{\mathbf{1}}{\sqrt{n}} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right).$$

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Let G be a d-regular graph.

- $\lambda_1 = d$ and the corresponding eigenvectors is

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- The graph is connected if and only if $\lambda_1 > \lambda 2$.
- The graph is bipartite if and only if $\lambda_1 = -\lambda_n$.

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Remark. *We actually proved that*

$$d = \lambda_1 \geq \ldots \geq \lambda_n \geq -d.$$

Expansion as Spectrum Gap

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Theorem

Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}.$$

Note $d - \lambda_2 = \lambda_1 - \lambda_2$ is the spectral gap of G.

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Lemma

Let $G = (V, E)$ be a d-regular graph with n-vertices. Then for all $S, T \subseteq V$,

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda\sqrt{|S||T|}.$$

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$$\begin{aligned} & \text{the number of edges between } u \text{ and } v \text{ in } E' \\ = \quad & \text{the number of length-2 walks between } u \text{ and } v \text{ in } G. \end{aligned}$$

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It follows $\lambda(G^2) = \lambda_2(G^2)$.

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Then

$$\begin{aligned} |E(S,T)| = 1_S A 1_T &= \left(\sum_{i \in [n]} \alpha_i v_i\right) A \left(\sum_{i \in [n]} \beta_i v_i\right) = \sum_{i,j \in [n]} (\alpha_i v_i) A (\beta_j v_j) \\ &= \sum_{i,j \in [n]} \lambda_j \alpha_i \beta_j \langle v_i, v_j \rangle = \sum_{i \in [n]} \lambda_i \alpha_i \beta_i \langle v_i, v_i \rangle = \sum_{i \in [n]} \lambda_i \alpha_i \beta_i. \end{aligned}$$

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Note

$$\alpha_1 = \left\langle 1_S, \frac{\mathbf{1}}{\sqrt{n}} \right\rangle = \frac{|S|}{\sqrt{n}}, \qquad \beta_1 = \left\langle 1_T, \frac{\mathbf{1}}{\sqrt{n}} \right\rangle = \frac{|T|}{\sqrt{n}}, \quad \text{and } \lambda_1 = d.$$

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Then

$$\big|E(S,T)\big| = \sum_{i \in [n]} \lambda_i \alpha_i \beta_i = \frac{d|S||T|}{n} + \sum_{2 \le i \le n} \lambda \alpha_i \beta_i$$

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Finally by **Cauchy-Schwartz**:

$$\left|\big|E(S,T)\big| - \frac{d|S||T|}{n}\right| \le \lambda \|\alpha\|_2 \|\beta\|_2 = \lambda \|1_S\|_2 \|1_T\|_2 = \lambda\sqrt{|S||T|}.$$

$\square$