Expander Graphs and Their Applications (XIII)

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Review of the Previous Lecture

Constraint Graph

Definition

$G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ is a constraint graph, if

1. (V, E) is an undirected graph (possibly with selfloops and multi-edges), i.e., *the underlying graph* of G.
2. The set V is also viewed as a set of variables assuming values over alphabet Σ.
3. Each edge $e \in E$ carries a constraint $c(e) \subseteq \Sigma^2$ and $\mathscr{C} = \{c(e) \mid e \in E\}$. A constraint $c(e)$ is said to be satisfied by (a, b) if $(a, b) \in c(e)$.

Remark.

- *The above definition is the rephrase of a CSP with each constraint being binary.*
- *Sometimes, we also use G to refer to (V, E).*

unsat

An assignment is a mapping $\sigma : V \to \Sigma$.

For a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ and an assignment σ:

$$\underline{\mathrm{unsat}_\sigma(G)} := \Pr_{\substack{e \in E \text{ with} \\ \text{endvertices } u \text{ and } v}} \big[(\sigma(u), \sigma(v)) \notin c(e) \big]$$

Then

$$\underline{\mathrm{unsat}(G)} := \min_\sigma \mathrm{unsat}_\sigma(G)$$

We have already seen:

Theorem

Given a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ with $|\Sigma| = 3$, it is NP-*hard to decide whether* $\mathrm{unsat}(G) = 0$.

Main Theorem

Definition

For every constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$, let

$$\underline{\mathrm{size}(G)} := |V| + |E|.$$

Theorem

There exists a Σ_0 *such that the following hold.*

For every finite alphabet Σ *there exist two constants* $C_\Sigma > 0$ *and* $0 < \alpha_\Sigma < 1$ *and a polynomial time algorithm* $\mathbb{A}_\Sigma$ *such that for every constraint graph* G*, the algorithm* $\mathbb{A}_\Sigma$ *computes a constraint graph* G' *such that*

- $\mathrm{size}(G') \leq C_\Sigma \cdot \mathrm{size}(G)$.
- *If* $\mathrm{unsat}(G) = 0$, *then* $\mathrm{unsat}(G') = 0$.
- *If* $\mathrm{unsat}(G) \neq 0$, *then* $\mathrm{unsat}(G') \geq \min(2 \cdot \mathrm{unsat}(G), \alpha_\Sigma)$.

Graph Powering

Fix some $d \in \mathbb{N}$.

Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph with (V, E) *being d-regular*, and $t \in \mathbb{N}$.

Recall, a sequence $(u_0, u_1, \ldots, u_t)$ is a $\underline{t\text{-step walk}}$ in G if there is an edge between u_{i-1} and u_i for all $i \in [t]$.

Then we will define the following t-th power of G

$$\underline{G^t} := \left\langle (V, E^t), \Sigma^{d^{\lceil t/2 \rceil}}, \mathscr{C}^t \right\rangle$$

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1. The vertices of G^t are the same as G. The number of edges in E^t between u and v is the number of t-step walks from u to v in G.
2. The alphabet of G^t is $\Sigma^{d^{\lceil t/2 \rceil}}$: Let

   $$\underline{\Gamma(u)} := \left\{ u' \in V \mid u = u_0, u_1, \ldots, u_{\lceil t/2 \rceil} = u' \text{ is a walk in } G \right\}.$$

   Then $|\Gamma(u)| \le d^{\lceil t/2 \rceil}$ and *by choosing some canonical order*, a value $a \in \Sigma^{d^{\lceil t/2 \rceil}}$ can be interpreted as an assignment $a : \Gamma(u) \to \Sigma$. One might think of this value as describing u's opinion of its neighbor's values.
3. The constraint $C(e)$ associated with an edge $e \in E^t$ with end vertices u and v contains those pairs $a, b \in \Sigma^{d^{\lceil t/2 \rceil}}$ if: There is an assignment $\sigma : \Gamma(u) \cup \Gamma(v) \to \Sigma$ that satisfies every constraint $c(e) \in \mathscr{C}$ where $e \in E \cap \big(\Gamma(u) \times \Gamma(v)\big)$, and such that

   $$\text{for all } u' \in \Gamma(u) \text{ and } v' \in \Gamma(v), \quad \sigma(u') = a_{u'} \text{ and } \sigma(v') = b_{v'}$$

   where $a_{u'}$ is the value a assigns u', and $b_{v'}$ the value b assigns to v'.

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Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d-regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$,

$$\text{unsat}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}(G), \frac{1}{t}\right).$$

Preprocessing Lemma

Lemma

There exist constants $0 < \lambda < d$ *and* $\beta_1 > 0$ *such that any constraint graph* G *can be transferred into a constraint graph* $G' := \underline{\text{prep}(G)}$ *such that*

- G' *is* d*-regular with selfloops and* $\lambda(G') \leq \lambda < d$.
- G' *has the same alphabet as* G *and* $\text{size}(G') = O(\text{size}(G))$.
- $\beta_1 \cdot \text{unsat}(G) \leq \text{unsat}(G') \leq \text{unsat}(G)$.

Alphabet Reduction

Lemma

There exist a constant $\beta_3 > 0$, an alphabet Σ_0, and a linear time algorithm $\mathbb{C}$ such that for every constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ the algorithm $\mathbb{C}$ computes a constraint graph G' over the alphabet Σ_0 such that

- $\text{size}(G') \leq c_\Sigma \cdot \text{size}(G)$, *where c_Σ is a constant only depending on Σ.*
- *and* $\beta_3 \cdot \text{unsat}(G) \leq \text{unsat}(G') \leq \text{unsat}(G)$.

Back to Expander Graphs

Construction of Expander Graphs

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There exist $d_0 \in \mathbb{N}$ and $h_0 > 0$ such that there is a polynomial time constructible family $\{X_n\}_{n \in \mathbb{N}}$ of d_0-regular graphs X_n on n vertices with $h(X_n) \geq h_0$.

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Using Zig-Zag product, we can construct $\{G_k\}_{k \in \mathbb{N}}$ such that:

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Every graph G_k is a $(d^{4k}, d^2, 1/2)$-graph for all $n \in \mathbb{N}$.

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Recall:

Theorem

Let G be a d-regular graph with second largest eigenvalue λ_1 and expansion ratio $h(G)$. Then

$$\lambda_1 \leq d - \frac{h(G)^2}{2d}.$$

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Corollary

There exist $d_0' \in \mathbb{N}$ and $0 < \lambda_0 < d_0'$ such that there is a polynomial time constructible family $\{X_n\}_{n \in \mathbb{N}}$ of d_0'-regular graphs X_n on n vertices with $\lambda(X_n) \leq \lambda_0$.

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Proof.

Take the d_0-regular graphs X_n from the previous lemma with $h(X_n) \geq h_0$. Add d_0 selfloops to each vertex. Take $d_0' := 2d_0$ and $\lambda_0 = d_0' - (h_0)^2/d_0'$. □

Preprocessing

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   Finally let $E' := E_1 \cup E_2$.
3. The constraints are $\mathscr{C}' := \{c(e')\}_{e' \in E'}$ with
   - If $e' \in E_1$ then $c(e') := \{(a, a) \mid a \in \Sigma\}$.
   - If $e' \in E_2$ has end vertices (v, e) and (v', e) then $c(e') := c(e) \in \mathscr{C}$.

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$$c \cdot \mathrm{unsat}(G) \leq \mathrm{unsat}(G') \leq \mathrm{unsat}(G)$$

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for some constant $c > 0$.

Moreover, for every assignment $\sigma' : V' \to \Sigma$ *let* $\sigma : V \to \Sigma$ *be defined according to the plurality value, i.e.,* $\sigma(v) := a$ *such that*

$$\Pr_{(v,e)\in[v]}[\sigma'(v,e) = a] \geq \Pr_{(v,e)\in[v]}[\sigma'(v,e) = a'] \text{ for all } a' \in \Sigma.$$

Then

$$c \cdot \text{unsat}_{\sigma}(G) \leq \text{unsat}_{\sigma'}(G').$$

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Thus

$$\mathrm{unsat}(G') \leq \mathrm{unsat}_{\sigma'}(G') = \frac{\mathrm{unsat}(G) \cdot |E_2|}{|E_1| + |E_2|} \leq \mathrm{unsat}(G).$$

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Fix an assignment $\sigma' : V' \to E$ and let $\sigma : V \to E$ be defined according to plurality value.

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$$|E'| = |E_1| + |E_2| \le d_0 \cdot |V'| + |E| \le 2 \cdot d_0 \cdot |E| + |E| \le 2 \cdot d \cdot |E|,$$

where the first inequality reaches equality when every edge in the expanders X_v is a selfloop, and the second inequality reaches equality if every edge in G is not a selfloop.

Fix an assignment $\sigma' : V' \to E$ and let $\sigma : V \to E$ be defined according to plurality value. Let

$$F := \{e \in E \mid \sigma \text{ violates } c(e) \in \mathscr{C}\}$$
$$F' := \{e \in E' \mid \sigma' \text{ violates } c(e) \in \mathscr{C}'\}$$

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$$\alpha := \frac{|F|}{|E|} = \text{unsat}_\sigma(G), \quad \text{we have } |F'| + |S| \geq |F| = \alpha \cdot |E|.$$

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$prep_2$

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Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph. The constraint graph $\underline{\mathrm{prep}_2(G)} := \langle (V, E'), \Sigma, \mathscr{C}' \rangle$ is defined as follows.

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   - If $e' \in E$ then $c(e')$ remains the same.
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prep_2 (cont'd)

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where I is the identity matrix.

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So the fraction of unsatisfied constrains cannot increase, and *drops by at most c'*. ∎

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There exist constants $0 < \lambda < \bar{d}$ *and* $\beta_1 > 0$ *such that any constraint graph* G *can be transferred into a constraint graph* $G' := \underline{\text{prep}(G)}$ *such that*

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Proof.

Let $G' := \text{prep}_2(\text{prep}_1(G))$, choose

$$\beta_1 := c \cdot \frac{d}{d + d_0' + 1}.$$

and $\bar{d} := d + d_0' + 1$

□