Expander Graphs and Their Applications (XIII)

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Review of the Previous Lecture

Constraint Graph

Definition

$G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ is a constraint graph, if

1. (V, E) is an undirected graph (possibly with selfloops and multi-edges), i.e., *the underlying graph* of G.
2. The set V is also viewed as a set of variables assuming values over alphabet Σ.
3. Each edge $e \in E$ carries a constraint $c(e) \subseteq \Sigma^2$ and $\mathscr{C} = \{c(e) \mid e \in E\}$. A constraint $c(e)$ is said to be satisfied by (a, b) if $(a, b) \in c(e)$.

Remark.

- *The above definition is the rephrase of a CSP with each constraint being binary.*
- *Sometimes, we also use G to refer to (V, E).*

unsat

An assignment is a mapping $\sigma : V \to \Sigma$.

For a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ and an assignment σ:

$$\underline{\text{unsat}_\sigma(G)} := \Pr_{\substack{e \in E \text{ with} \\ \text{endvertices } u \text{ and } v}} \left[(\sigma(u), \sigma(v)) \notin c(e) \right]$$

Then

$$\underline{\text{unsat}(G)} := \min_\sigma \text{unsat}_\sigma(G)$$

We have already seen:

Theorem

Given a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ with $|\Sigma| = 3$, it is NP-*hard to decide whether* $\text{unsat}(G) = 0$.

Main Theorem

Definition

For every constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$, let

$$\text{size}(G) := |V| + |E|.$$

Theorem

There exists a Σ_0 *such that the following hold.*

For every finite alphabet Σ *there exist two constants* $C_\Sigma > 0$ *and* $0 < \alpha_\Sigma < 1$ *and a polynomial time algorithm* $\mathbb{A}_\Sigma$ *such that for every constraint graph* G*, the algorithm* $\mathbb{A}_\Sigma$ *computes a constraint graph* G' *such that*

- $\text{size}(G') \leq C_\Sigma \cdot \text{size}(G)$.
- *If* $\text{unsat}(G) = 0$, *then* $\text{unsat}(G') = 0$.
- *If* $\text{unsat}(G) \neq 0$, *then* $\text{unsat}(G') \geq \min(2 \cdot \text{unsat}(G), \alpha_\Sigma)$.

Graph Powering

Fix some $d \in \mathbb{N}$.

Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph with *(V, E) being d-regular*, and $t \in \mathbb{N}$.

Recall, a sequence $(u_0, u_1, \ldots, u_t)$ is a t-step walk in G if there is an edge between u_{i-1} and u_i for all $i \in [t]$.

Then we will define the following t-th power of G

$$\underline{G^t} := \left\langle (V, E^t), \Sigma^{d^{\lceil t/2 \rceil}}, \mathscr{C}^t \right\rangle$$

$G^t := \left\langle (V, E^t), \Sigma^{d^{\lceil t/2 \rceil}}, \mathscr{C}^t \right\rangle$

1. The vertices of G^t are the same as G. The number of edges in E^t between u and v is the number of t-step walks from u to v in G.
2. The alphabet of G^t is $\Sigma^{d^{\lceil t/2 \rceil}}$: Let

$$\underline{\Gamma(u)} := \left\{ u' \in V \mid u = u_0, u_1, \ldots, u_{\lceil t/2 \rceil} = u' \text{ is a walk in } G \right\}.$$

   Then $|\Gamma(u)| \leq d^{\lceil t/2 \rceil}$ and *by choosing some canonical order*, a value $a \in \Sigma^{d^{\lceil t/2 \rceil}}$ can be interpreted as an assignment $a : \Gamma(u) \to \Sigma$. One might think of this value as describing u's opinion of its neighbor's values.
3. The constraint $C(e)$ associated with an edge $e \in E^t$ with end vertices u and v contains those pairs $a, b \in \Sigma^{d^{\lceil t/2 \rceil}}$ if: There is an assignment $\sigma : \Gamma(u) \cup \Gamma(v) \to \Sigma$ that satisfies every constraint $c(e) \in \mathscr{C}$ where $e \in E \cap \big(\Gamma(u) \times \Gamma(v)\big)$, and such that

$$\text{for all } u' \in \Gamma(u) \text{ and } v' \in \Gamma(v), \quad \sigma(u') = a_{u'} \text{ and } \sigma(v') = b_{v'}$$

   where $a_{u'}$ is the value a assigns u', and $b_{v'}$ the value b assigns to v'.

Amplification Lemma

Clearly $\text{unsat}(G) = 0$ implies $\text{unsat}(G^t) = 0$. What about the other direction?

Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d-regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$,

$$\text{unsat}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}(G), \frac{1}{t}\right).$$

Preprocessing Lemma

Lemma

There exist constants $0 < \lambda < d$ *and* $\beta_1 > 0$ *such that any constraint graph* G *can be transferred into a constraint graph* $G' := \underline{\mathrm{prep}(G)}$ *such that*

- G' *is* d*-regular with selfloops and* $\lambda(G') \leq \lambda < d$.
- G' *has the same alphabet as* G *and* $\mathrm{size}(G') = O(\mathrm{size}(G))$.
- $\beta_1 \cdot \mathrm{unsat}(G) \leq \mathrm{unsat}(G') \leq \mathrm{unsat}(G)$.

Alphabet Reduction

Lemma

There exist a constant $\beta_3 > 0$, an alphabet Σ_0, and a linear time algorithm $\mathbb{C}$ such that for every constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ the algorithm $\mathbb{C}$ computes a constraint graph G' over the alphabet Σ_0 such that

- $\text{size}(G') \leq c_\Sigma \cdot \text{size}(G)$, *where c_Σ is a constant only depending on Σ.*
- *and* $\beta_3 \cdot \text{unsat}(G) \leq \text{unsat}(G') \leq \text{unsat}(G)$.

Back to Expander Graphs

Lemma

There exist $d_0 \in \mathbb{N}$ and $h_0 > 0$ such that there is a polynomial time constructible family $\{X_n\}_{n\in\mathbb{N}}$ of d_0-regular graphs X_n on n vertices with $h(X_n) \geq h_0$.

Using Zig-Zag product, we can construct $\{G_k\}_{k\in\mathbb{N}}$ such that:

Theorem

Every graph G_k is a $(d^{4k}, d^2, 1/2)$-graph for all $n \in \mathbb{N}$.

Proof of the lemma. If $n = d^{4k}$ for some $k \in \mathbb{N}$, then take $X_n := G_k$ and *add appropriate number of new selfloops to each vertex*.

If $d^{4k-4} < n < d^{4k}$, then let $m := d^{4k} - n < (d^4 - 1) \cdot n$. Take G_k and

- *merge vertices to decrease the number of vertices to n*;
- add appropriate number of selfloops to every vertices.

□

Construction of Expander Graphs (cont'd)

Recall:

Theorem

Let G be a d-regular graph with second largest eigenvalue λ_1 and expansion ratio $h(G)$. Then

$$\lambda_1 \leq d - \frac{h(G)^2}{2d}.$$

Corollary

There exist $d_0' \in \mathbb{N}$ and $0 < \lambda_0 < d_0'$ such that there is a polynomial time constructible family $\{X_n\}_{n\in\mathbb{N}}$ of d_0'-regular graphs X_n on n vertices with $\lambda(X_n) \leq \lambda_0$.

Proof.

Take the d_0-regular graphs X_n from the previous lemma with $h(X_n) \geq h_0$. Add d_0 selfloops to each vertex. Take $d_0' := 2d_0$ and $\lambda_0 = d_0' - (h_0)^2/d_0'$. □

Preprocessing

prep_1

Definition

Let $d_0 \in \mathbb{N}$ be the constant from the previous first lemma for the existence of expanders. Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph. The constraint graph $\text{prep}_1(G) := \langle (V', E'), \Sigma, \mathscr{C}' \rangle$ is defined as follows.

1. For each $v \in V$ let $[v] := \{(v, e) \mid e \in E \text{ is incident to } v\}$, and set $V' := \bigcup_{v \in V} [v]$.
2. For each $v \in V$ let X_v be a d_0-regular graph on vertex set $[v]$ and edge expansion at least h_0. Let $E_1 := \bigcup_{c \in V} E(X_v)$ and set

   $$E_2 := \{\text{an edge between } (v, e) \text{ and } (v', e) \mid \text{an edge } e \in E \text{ between } v \text{ and } v'\}$$

   Finally let $E' := E_1 \cup E_2$.
3. The constraints are $\mathscr{C}' := \{c(e')\}_{e' \in E'}$ with
   - If $e' \in E_1$ then $c(e') := \{(a, a) \mid a \in \Sigma\}$.
   - If $e' \in E_2$ has end vertices (v, e) and (v', e) then $c(e') := c(e) \in \mathscr{C}$.

prep_1 (cont'd)

Lemma

Let $G = \langle (V,E), \Sigma, \mathscr{C} \rangle$ be a constraint graph and $G' = \langle (V',E'), \Sigma, \mathscr{C}' \rangle$ $:= \text{prep}_1(G)$. Then G' is a $d := (d_0 + 1)$-regular constraint graph such that $|V'| \leq 2|E|$ and

$$c \cdot \text{unsat}(G) \leq \text{unsat}(G') \leq \text{unsat}(G)$$

for some constant $c > 0$.

Moreover, for every assignment $\sigma' : V' \to \Sigma$ let $\sigma : V \to \Sigma$ be defined according to the plurality value, i.e., $\sigma(v) := a$ such that

$$\Pr_{(v,e)\in[v]}[\sigma'(v,e) = a] \geq \Pr_{(v,e)\in[v]}[\sigma'(v,e) = a'] \text{ for all } a' \in \Sigma.$$

Then

$$c \cdot \text{unsat}_\sigma(G) \leq \text{unsat}_{\sigma'}(G').$$

Proof.

$\text{unsat}(G') \leq \text{unsat}(G)$: An *optimal* assignment $\sigma : V \to \Sigma$, i.e.,

$$\text{unsat}_\sigma(G) = \text{unsat}(G)$$

can be extended to an assignment $\sigma' : V' \to E$ by $\sigma'(v, e) := \sigma(v)$.
σ' does not violate the constraints $c(e')$ for $e' \in E_1$, as they are equality constraints. And it violates exactly $\text{unsat}_\sigma(G) \cdot |E_2| = \text{unsat}(G) \cdot |E_2|$ constraints corresponding to E_2.
Thus

$$\text{unsat}(G') \leq \text{unsat}_{\sigma'}(G') = \frac{\text{unsat}(G) \cdot |E_2|}{|E_1| + |E_2|} \leq \text{unsat}(G).$$

Proof. (cont'd)

$c \cdot \text{unsat}_\sigma(G) \le \text{unsat}_{\sigma'}(G')$: First observe that $|E'| \le 2 \cdot d \cdot |E|$:

$$\left|E'\right| = |E_1| + |E_2| \le d_0 \cdot |V'| + |E| \le 2 \cdot d_0 \cdot |E| + |E| \le 2 \cdot d \cdot |E|,$$

where the first inequality reaches equality when every edge in the expanders X_v is a selfloop, and the second inequality reaches equality if every edge in G is not a selfloop.

Fix an assignment $\sigma' : V' \to E$ and let $\sigma : V \to E$ be defined according to plurality value. Let

$$F := \left\{e \in E \mid \sigma \text{ violates } c(e) \in \mathscr{C}\right\}$$
$$F' := \left\{e \in E' \mid \sigma' \text{ violates } c(e) \in \mathscr{C}'\right\}$$

Proof. (cont'd)

Let $S \subseteq V'$ be the set of vertices of G' whose value disagrees with the the plurality:

$$S := \bigcup_{v \in V} \{(v, e) \in [v] \mid \sigma'(v, e) \neq \sigma(v)\}.$$

Let $e \in F$ with end vertices v and v'. Then the edge $e' \in E$ with end vertices (v, e) and (v', e) *either belongs to F'*, or *has at least one end vertex in S*. Hence for

$$\alpha := \frac{|F|}{|E|} = \text{unsat}_\sigma(G), \quad \text{we have } |F'| + |S| \geq |F| = \alpha \cdot |E|.$$

Proof. (cont'd)

- If $|F'| \geq \alpha/2 \cdot |E|$ we are done since $\alpha/2 \cdot |E| \geq \alpha/(4 \cdot d) \cdot |E'|$ and so

$$\text{unsat}_{\sigma'}(G') \geq \text{unsat}_{\sigma}(G)/4 \cdot d.$$

- If $|F'| < \alpha/2 \cdot |E|$, by $|F'| + |S| \geq \alpha \cdot |E|$, then $|S| \geq \alpha/2 \cdot |E|$.
  For every $v \in V$ let $S^v := [v] \cap S$, S^v is the *disjoint union* of all

$$\underline{S^v_a} := \{(v, e) \in S^v \mid \sigma'(v, e) = a\} \qquad \text{for all } a \in \Sigma.$$

  Then $|S^v_a| \leq \big|[v]\big|/2$. Recall X_v is an expander with expansion ratio h_0, so

$$\big|E(S^v_a, [v] \setminus S^v_a)\big| \geq h_0 \cdot |S^v_a|.$$

  All the edges in X_v leaving S^v_a carry equality constraints which σ' violates.
  There are at least

$$\frac{h_0}{2} \cdot \sum_v \big|S \cap [v]\big| = \frac{h_0}{2} \cdot |S| \geq \frac{\alpha \cdot h_0}{4} \cdot |E|$$

  edges that σ' violates. $\text{unsat}_{\sigma'}(G') \geq h_0/(4 \cdot d) \cdot \text{unsat}_{\sigma}(G)$, by $|E| \geq |E'|/d$. We can take $c := \min\{1/(4 \cdot d), h_0/(4 \cdot d)\}$. □

Definition

Let $d_0', \lambda_0 \in \mathbb{N}$ be the constants from the previous second lemma for the existence of expanders.

Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph. The constraint graph $\text{prep}_2(G) := \langle (V, E'), \Sigma, \mathscr{C}' \rangle$ is defined as follows.

1. The vertices remain the same.
2. Let X be a d_0'-regular graph on vertex set V and edge set E_1 with $\lambda(X) \le \lambda_0 < d_0'$. Let $E_2 := \{\text{a new selfloop on } v \mid v \in V\}$ and finally let $E' := E \cup E_1 \cup E_2$.
3. The constraints are $\mathscr{C}' := \{c(e')\}_{e' \in E'}$ with
   - If $e' \in E$ then $c(e')$ remains the same.
   - If $e' \in E_1 \cup E_2$, then $c(e') := \{(a, b) \mid a, b \in \Sigma\}$.

Lemma

Let G be a d-regular constraint graph G and $G' := \text{prep}_2(G)$. Then G' has the following properties.

- *G' is $(d + d_0' + 1)$-regular, has a selfloop on every vertex, and* $\lambda(G') \leq d + \lambda_0 + 1 < \deg(G')$.
- *For every $\sigma : V \to \Sigma$,*

$$\frac{d}{d + d_0' + 1} \cdot \text{unsat}_\sigma(G) \leq \text{unsat}_\sigma(G') \leq \text{unsat}_\sigma(G).$$

Proof. Let A_G be the adjacency matrix of G. Then

$$\lambda(G) = \max_{\|\vec{x}\|_2=1, \vec{x}\perp\mathbf{1}} \left| \langle \vec{x}, A_G\vec{x}\rangle \right|.$$

Let A_X, $A_{G'}$ be the adjacency matrices of X and G', respectively. Then

$$A_{G'} = A_G + I + A_X,$$

where I is the identity matrix.

Proof. (cont'd)

Then,

$$\begin{aligned}\lambda(G') &= \max_{\|\vec{x}\|_2=1, \vec{x}\perp\mathbf{1}} \left| \langle \vec{x}, A_{G'}\vec{x}\rangle \right| \\ &\leq \max_{\|\vec{x}\|_2=1, \vec{x}\perp\mathbf{1}} \left| \langle \vec{x}, A_{G}\vec{x}\rangle \right| + \max_{\|\vec{x}\|_2=1, \vec{x}\perp\mathbf{1}} \left| \langle \vec{x}, I\vec{x}\rangle \right| + \max_{\|\vec{x}\|_2=1, \vec{x}\perp\mathbf{1}} \left| \langle \vec{x}, A_{X}\vec{x}\rangle \right| \\ &= \lambda(G) + \lambda(I) + \lambda(X) \leq d + 1 + \lambda_0.\end{aligned}$$

Finally, let $\sigma : V \to \Sigma$. All new edges are always satisfied by every assignment. And we increase the number of edges by at most a factor

$$c' := \frac{d + d_0' + 1}{d}.$$

So the fraction of unsatisfied constrains cannot increase, and *drops by at most* c'. □

Preprocessing Lemma

Lemma

There exist constants $0 < \lambda < \bar{d}$ *and* $\beta_1 > 0$ *such that any constraint graph* G *can be transferred into a constraint graph* $G' := \underline{\text{prep}(G)}$ *such that*

- G' *is* $\bar{d}$*-regular with selfloops and* $\lambda(G') \leq \lambda < \bar{d}$.
- G' *has the same alphabet as* G *and* $\text{size}(G') = O(\text{size}(G))$.
- $\beta_1 \cdot \text{unsat}(G) \leq \text{unsat}(G') \leq \text{unsat}(G)$.

Proof.

Let $G' := \text{prep}_2(\text{prep}_1(G))$, choose

$$\beta_1 := c \cdot \frac{d}{d + d_0' + 1}.$$

and $\bar{d} := d + d_0' + 1$

□