

# Expander Graphs and Their Applications (XVI)

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## Review of the Previous Lecture

## Amplification Lemma (stronger version)

### Lemma

Let  $0 < \lambda < d$  and  $|\Sigma|$  be constants. Then there exists a constant  $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$  such that for every  $t \in \mathbb{N}$  and for every  $d$ -regular constraint graph  $G$  with a selfloop on each vertex and  $\lambda(G) \leq \lambda$ , the following hold.

For every  $\vec{\sigma} : V \rightarrow \Sigma^{d^{\lceil t/2 \rceil}}$  let  $\sigma : V \rightarrow \Sigma$  be defined according to “popular opinion” by setting, for each  $v \in V$ ,  $\sigma(v) := a$  such that

$\Pr[\text{a random } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \text{ reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a],$

where  $\vec{\sigma}(w)_v \in \Sigma$  denotes the restriction of  $\vec{\sigma}(w)$  to  $v$ , is maximized over all  $a \in \Sigma$ .

Then

$$\text{unsat}_{\vec{\sigma}}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right).$$

## Proof of the Amplification Lemma

Let  $\vec{\sigma} : V \rightarrow \Sigma^{d^{\lceil t/2 \rceil}}$  be an assignment for  $G^t$ . Define the assignment  $\sigma : V \rightarrow \Sigma$  as before, i.e., according to “popular opinion.”

For every  $v \in V$ , let  $X_v$  be a random variable such that for every  $a \in \Sigma$

$$\Pr[X_v = a] = \Pr[\text{a } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \\ \text{reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a].$$

Then for every  $a \in \Sigma$ ,

$$\Pr[X_v = \sigma(v)] \geq \Pr[X_v = a].$$

Hence,

$$\Pr[X_v = \sigma(v)] \geq \frac{1}{|\Sigma|}.$$

## Proof of the Amplification Lemma (cont'd)

Let

$$F := \begin{cases} \{e \in E \mid \sigma \text{ violates } e\} & \text{if } \text{unsat}_\sigma(G) < 1/t \\ \text{an arbitrary subset of the above } \{\dots\} & \\ \text{with } |F| = \lfloor |E|/t \rfloor & \text{otherwise.} \end{cases}$$

Then

$$\Omega\left(\frac{|F|}{|E|}\right) = \min\left(\text{unsat}_\sigma(G), \frac{1}{t}\right)$$

**From now on, we fix  $\vec{\sigma}$ ,  $\sigma$ , and  $F$ .**

## Proof of the Amplification Lemma (cont'd)

Let  $\mathbf{E} := E(G^t) = E^t$ . Recall there is a one-to-one correspondence between every edge  $\mathbf{e} \in \mathbf{E}$  and every walk of length  $t$  in  $G$ .

With some abuse of notation we write  $\mathbf{e} = (v_0, v_1, \dots, v_t)$  where  $(v_{i-1}, v_i) \in E$  for all  $i \in [t]$ .

### Definition

A walk  $\mathbf{e} = (v_0, v_1, \dots, v_t)$  is hit by its  $i$ -th edge if

1.  $(v_{i-1}, v_i) \in F$ , and
2. Both  $\vec{\sigma}(v_0)_{v_{i-1}} = \sigma(v_{i-1})$  and  $\vec{\sigma}(v_t)_{v_i} = \sigma(v_i)$ .

### Remark.

- $(v_{i-1}, v_i) \in F$  means that the edge  $(v_{i-1}, v_i)$  rejects  $\sigma$ .
- By 2,  $v_0$  has the major opinion of  $v_{i-1}$  in  $G^t$ , (which implies that *there is a  $\lfloor t/2 \rfloor$ -step walk from  $v_0$  to  $v_{i-1}$* ). Similarly,  $v_t$  has the majority opinion of  $v_i$ .

*Hence,  $(v_0, v_t)$  rejects  $\vec{\sigma}$  by our definition of  $G^t$ .*

## Proof of the Amplification Lemma (cont'd)

Let

$$I := \left\{ \frac{t}{2} - \sqrt{\frac{t}{2}} < i \leq \frac{t}{2} + \sqrt{\frac{t}{2}} \right\} \subseteq \mathbb{N}$$

be the set of “middle” indices. For each walk  $\mathbf{e}$ , we define

$$N(\mathbf{e}) := |\{i \in I \mid \mathbf{e} \text{ is hit by its } i\text{-th edge}\}|.$$

$N(\mathbf{e}) > 0$  implies that  $\mathbf{e}$  rejects  $\vec{\sigma}$ .

Thus  $\Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \Pr_{\mathbf{e}}[\mathbf{e} \text{ rejects } \vec{\sigma}] = \text{unsat}_{\vec{\sigma}}(G^t)$ .

We will prove

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

Combined with  $\Omega(|F|/|E|) = \min(\text{unsat}_{\sigma}(G), 1/t)$ ,

$$\Omega(\sqrt{t}) \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right) \leq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \text{unsat}_{\vec{\sigma}}(G^t).$$

## Proof of the Amplification Lemma (cont'd)

To show:

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

we will prove two lemmas.

Lemma

$$\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Lemma

$$\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$



## Proof of the Amplification Lemma (cont'd)

From probability theory:

### Lemma

For every *non-negative* random variable  $X$  which is not identically zero,

$$\Pr[X > 0] \geq \frac{\mathbb{E}^2[X]}{\mathbb{E}[X^2]}.$$

Proof.

$$\mathbb{E}[X] = \mathbb{E}[X \cdot \mathbf{1}_{X>0}] \leq \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\mathbb{E}[(\mathbf{1}_{X>0})^2]} = \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\Pr[X > 0]}.$$

□

Now by the previous lemmas

$$\Pr[N(\mathbf{e}) > 0] \geq \mathbb{E}^2[N(\mathbf{e})]/\mathbb{E}[(N(\mathbf{e}))^2] = \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

## Proof of $\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot |F|/|E|$

For a walk  $\mathbf{e}$ , let  $\mathbf{e}_i$  denote its  $i$ -th edge, i.e.,  $\mathbf{e}_i = (v_{i-1}, v_i)$ . To upper bound  $\mathbb{E}[N^2]$  we define a random variable

$$\underline{Z}(\mathbf{e}) := \{i \in I \mid \mathbf{e}_i \in F\},$$

i.e.,  $Z(\mathbf{e})$  counts how many times  $\mathbf{e}$  intersects  $F$  in the *middle* portion  $I$ .

$$0 \leq N(\mathbf{e}) \leq Z(\mathbf{e}) \quad \text{and} \quad \mathbb{E}[N^2] \leq [Z^2].$$

For every  $i \in I$ , let

$$\underline{Z}_i(\mathbf{e}) := 1 \iff \mathbf{e}_i \in F.$$

Therefore  $Z(\mathbf{e}) = \sum_{i \in I} Z_i(\mathbf{e})$ . And

$$\begin{aligned} \mathbb{E}_{\mathbf{e}}[Z(\mathbf{e})^2] &= \sum_{i,j \in I} \mathbb{E}_{\mathbf{e}}[Z_i(\mathbf{e}) \cdot Z_j(\mathbf{e})] \\ &= \sum_{i \in I} \mathbb{E}[Z_i] + 2 \cdot \sum_{i,j \in I, i < j} \mathbb{E}[Z_i \cdot Z_j] = |I| \cdot \frac{|F|}{|E|} + 2 \cdot \sum_{i,j \in I, i < j} \mathbb{E}[Z_i \cdot Z_j] \end{aligned}$$

## Proof of $\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

We will prove:

**Proposition** Let  $i, j \in I$  with  $i < j$  and  $F \subseteq E$ . Then

$$\mathbb{E}[Z_i \cdot Z_j] \leq \frac{|F|}{|E|} \cdot \left( \frac{|F|}{|E|} + \left( \frac{\lambda}{d} \right)^{j-i-1} \right).$$

By this proposition

$$\begin{aligned} \mathbb{E}_{\mathbf{e}}[Z(\mathbf{e})^2] &= |I| \cdot \frac{|F|}{|E|} + 2 \cdot \sum_{i, j \in I, i < j} \mathbb{E}[Z_i \cdot Z_j] \\ &\leq O(\sqrt{t}) \cdot \frac{|F|}{|E|} + 2 \cdot \frac{|F|}{|E|} \cdot \sum_{i, j \in I, i < j} \left( \frac{|F|}{|E|} + \left( \frac{\lambda}{d} \right)^{j-i} \right) \\ &\leq O(\sqrt{t}) \cdot \frac{|F|}{|E|} + 2 \cdot |I|^2 \cdot \left( \frac{|F|}{|E|} \right)^2 + 2 \cdot |I| \cdot \frac{|F|}{|E|} \cdot \sum_{i=1}^{\sqrt{2t}} (\lambda/d)^i \\ &= O(\sqrt{t}) \cdot \frac{|F|}{|E|}. \quad \square \end{aligned}$$

Proof of  $\mathbb{E}[Z_i \cdot Z_j] \leq |F|/|E| \cdot (|F|/|E| + (\lambda/d)^{j-i-1})$

Observe that  $Z_i, Z_j \in \{0, 1\}$  and

$$\Pr[Z_j = 1] = \Pr[Z_i = 1] = \frac{|F|}{|E|}.$$

Thus

$$\begin{aligned}\mathbb{E}[Z_i \cdot Z_j] &= \Pr[Z_i \cdot Z_j = 1] = \Pr[Z_i = 1] \cdot \Pr[Z_j = 1 | Z_i = 1] \\ &= \frac{|F|}{|E|} \cdot \Pr[Z_j = 1 | Z_i = 1].\end{aligned}$$

Proof of  $\mathbb{E}[Z_i \cdot Z_j] \leq |F|/|E| \cdot (|F|/|E| + (\lambda/d)^{j-i-1})$  (cont'd)

We will need the following result:

### Theorem

*Let  $G = (V, E)$  be a  $d$ -regular graph with  $\lambda(G) \leq \lambda$ . Let  $F \subseteq E$  be a set of edges without selfloops, and let  $K$  be the distribution on vertices induced by selecting a random edge in  $F$  and then a random end vertex.*

*The probability that a random walk that starts with distribution  $K$  takes the  $i + 1$ -th step in  $F$  is upper bounded by*

$$\frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i.$$

Proof of  $\mathbb{E}[Z_i \cdot Z_j] \leq |F|/|E| \cdot (|F|/|E| + (\lambda/d)^{j-i-1})$  (cont'd)

Assume first  $i = 1$  and  $j > i$ . Then by the previous theorem

$$\Pr_{\mathbf{e}} [Z_j(\mathbf{e}) = 1 | Z_1(\mathbf{e}) = 1] \leq \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-2}$$

If  $j > i > 1$ , we don't care where the random walk  $\mathbf{e}$  visited during its first  $i - 1$  steps, so they can be ignored.

Thus

$$\begin{aligned} \Pr_{|\mathbf{e}|=t} [Z_j(\mathbf{e}) = 1 | Z_i(\mathbf{e}) = 1] &= \Pr_{|\mathbf{e}'|=t-i+1} [Z_{j-i+1}(\mathbf{e}') = 1 | Z_1(\mathbf{e}') = 1] \\ &\leq \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-i-1}, \end{aligned}$$

again by the previous theorem.

□

## Theorem

Let  $G = (V, E)$  be a  $d$ -regular graph with  $\lambda(G) \leq \lambda$ . Let  $F \subseteq E$  be a set of edges without selfloops, and let  $K$  be the distribution on vertices induced by selecting a random edge in  $F$  and then a random end vertex.

The probability that a random walk that starts with distribution  $K$  takes the  $i + 1$ -th step in  $F$  is upper bounded by

$$\frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i.$$

## Proof.

Let  $B \subseteq V$  be the *support of  $K$* , i.e.,

$$\underline{B} := \{v \in V \mid K(v) > 0\}.$$

Let  $n := |V|$ . Recall  $\hat{A}$  is the normalized  $n \times n$  adjacency matrix of  $G$ . The first and second largest eigenvalues (in absolute value) of  $\hat{A}$  are  $\mathbf{1}$  and

$$\underline{\hat{\lambda}} := \frac{\lambda(G)}{d}.$$

Let  $\underline{\vec{x}}$  be the vector corresponding to the distribution  $K$ , i.e.,  $\vec{x} = (x_v)_{v \in V}$  with

$$x_v = \Pr_K[v] = \text{the fraction of edges touching } v \text{ that are in } F, \text{ divided by } 2.$$

Let  $\underline{y_v}$  be the probability that a random step from  $v$  is in  $F$ . Therefore

$$\vec{y} := (y_v)_{v \in V} = \frac{2 \cdot |F|}{d} \cdot \vec{x}.$$



## Proof. (cont'd)

The probability  $p$  equals the probability of landing in  $B$  after  $i$  steps, and then taking a step in  $F$ , i.e.,

$$p = \sum_{v \in B} y_v (\hat{A}^i \vec{x})_v = \sum_{v \in V} y_v (\hat{A}^i \vec{x})_v = \langle \vec{y}, \hat{A}^i \vec{x} \rangle.$$

Now write

$$\vec{x} = \vec{x}^\perp + \vec{x}^\parallel \quad \text{with } \vec{x}^\parallel := \mathbf{1}/n.$$

Recall  $\vec{x}^\perp$  is orthogonal to  $\vec{x}^\parallel$ . Thus,

$$\|\hat{A}^i \vec{x}^\perp\|_2 \leq \hat{\lambda}^i \|\vec{x}^\perp\|_2 \leq \hat{\lambda}^i \|\vec{x}\|_2.$$

Observe that  $\|\vec{x}\|_2^2 \leq (\sum_{v \in V} |x_v|) \cdot (\max_{v \in V} |x_v|) \leq \max_{v \in V} |x_v| \leq d/(2 \cdot |F|)$ .

$$\langle \vec{y}, \hat{A}^i \vec{x}^\perp \rangle \leq \|\vec{y}\|_2 \cdot \|\hat{A}^i \vec{x}^\perp\|_2 \leq \frac{2 \cdot |F|}{d} \cdot \|\vec{x}\|_2 \cdot \hat{\lambda}^i \cdot \|\vec{x}\|_2 \leq \hat{\lambda}^i$$

Finally,

$$\langle \vec{y}, \hat{A}^i \vec{x} \rangle \leq \langle \vec{y}, \hat{A}^i \vec{x}^\parallel \rangle + \langle \vec{y}, \hat{A}^i \vec{x}^\perp \rangle \leq \frac{2 \cdot |F|}{d \cdot n} + \hat{\lambda}^i \leq \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i. \quad \square$$