Expander Graphs and Their Applications (XIX)

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Review of the Previous Lecture

Assignment Tester

Definition

An Assignment Tester with alphabet Σ_0 and reject probability $\varepsilon > 0$ is an algorithm $\mathbb{P}$ whose input is a circuit Φ over Boolean variables in the set X, and whose output is a constraint graph $G = \langle (V, E), \Sigma_0, \mathscr{C} \rangle$ such that $X \subseteq V$ and such that the following hold.

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Let $V' := V \setminus X$ and let $a : X \to \{0, 1\}$ be an assignment.

- **(Completeness)** if a satisfies Φ, then there exists some $b : V' \to \Sigma_0$ such that $\mathrm{unsat}_{a \cup b}(G) = 0$.
- **(Completeness)** if a does not satisfy Φ, then for all $b : V' \to \Sigma_0$

$$\mathrm{unsat}_{a \cup b}(G) \geq \varepsilon \cdot \mathrm{rdist}(a, s)$$

  for every $s : X \to \{0, 1\}$ that satisfies Φ.

Theorem

There is some $\varepsilon > 0$ and an explicit construction of an assignment tester $\mathbb{P}$ with alphabet $\Sigma_0 = \{0, 1\}^3$ and rejection probability ε.

Alphabet Reduction by Composition

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e is of relative distance $\rho > 0$ if for every $a_1, a_2 \in \Sigma$

$$\mathrm{rdist}(e(a_1, e_(a_2)) \leq \rho.$$

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2. **(Composition:)** Run the assignment tester $\mathbb{P}$ on each $\tilde{c}(e)$ which outputs $G_e = \langle (V_e, E_e), \Sigma_0, \mathscr{C}_e \rangle$. Assume, without loss of generality, E_e has the same cardinality for every $e \in E$. Finally, we define the constraint graph $G \circ \mathbb{P} := \langle (V', E'), \Sigma_0, \mathscr{C}' \rangle$ by

$$V' := \bigcup_{e \in E} V_e, \quad E' := \bigcup_{e \in E} E_e, \quad \mathscr{C}' := \bigcup_{e \in E} \mathscr{C}_e.$$

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Given any constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$, one can compute in time line in $\mathrm{size}(G)$, the constraint graph $G' = G \circ \mathbb{P}$, with $\mathrm{size}(G') = c(\mathbb{P}, |\Sigma|) \cdot \mathrm{size}(G)$, and

$$\beta_3 \cdot \mathrm{unsat}(G) \le \mathrm{unsat}(G') \le \mathrm{unsat}(G).$$

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As the size of the input to $\mathbb{P}$ is bounded by some constant only depending on $|\Sigma|$, the running time of $\mathbb{P}$ is bounded by a constant, and the size of the output is also bounded by some constant.
Therefore, the blowup factor depends only on $|\Sigma|$ and on $\mathbb{P}$, i.e.,

$$\text{size}(G') = c(\mathbb{P}, |\Sigma|) \cdot \text{size}(G).$$

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where $\sigma'([v])$ means the concatenation of $\sigma'(v)$ for all $y \in [v]$.

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It remains to define values for σ' on

$$\bigcup_{e=(u,v)\in E} \left(V_e \setminus \left([u] \cup [v]\right)\right)$$

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So if let a be the restriction of σ' to $[u] \cup [v]$, then there is some $b : V_e \setminus ([u] \cup [v]) \to \Sigma_0$ such that $\mathrm{unsat}_{a \cup b}(G_e) = 0$.

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Let $e = (u, v) \in F$. We will show that *at least β_3 fraction of the constraints of G_e is falsified by σ'.*

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□