Expander Graphs and Their Applications (XIX)

Yijia Chen
Shanghai Jiaotong University

Review of the Previous Lecture

Assignment Tester

Definition

An Assignment Tester with alphabet Σ_0 and reject probability $\varepsilon > 0$ is an algorithm $\mathbb{P}$ whose input is a circuit Φ over Boolean variables in the set X, and whose output is a constraint graph $G = \langle (V, E), \Sigma_0, \mathscr{C} \rangle$ such that $X \subseteq V$ and such that the following hold.
Let $V' := V \setminus X$ and let $a : X \to \{0,1\}$ be an assignment.

- **(Completeness)** if a satisfies Φ, then there exists some $b : V' \to \Sigma_0$ such that $\mathrm{unsat}_{a \cup b}(G) = 0$.
- **(Completeness)** if a does not satisfy Φ, then for all $b : V' \to \Sigma_0$

$$\mathrm{unsat}_{a \cup b}(G) \geq \varepsilon \cdot \mathrm{rdist}(a, s)$$

for every $s : X \to \{0,1\}$ that satisfies Φ.

Theorem

There is some $\varepsilon > 0$ and an explicit construction of an assignment tester $\mathbb{P}$ with alphabet $\Sigma_0 = \{0,1\}^3$ and rejection probability ε.

Alphabet Reduction by Composition

Composition

Let Σ be a finite alphabet and let $e : \Sigma \to \{0,1\}^\ell$ (an error correcting code).

e is of linear dimension if there is a constant $c > 0$ such that

$$\ell \leq c \cdot \log_2 |\Sigma|, \qquad \text{i.e.,} \quad \left|\{0,1\}^\ell\right| \leq |\Sigma|^c.$$

e is of relative distance $\rho > 0$ if for every $a_1, a_2 \in \Sigma$

$$\text{rdist}(e(a_1, e_(a_2)) \leq \rho.$$

Composition (cont'd)

Definition

Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph and let $\mathbb{P}$ be an assignment tester. Moreover, let $e : \Sigma \to \{0,1\}^{\ell}$ be an encoding with linear dimension and relative distance $\rho > 0$. The constraint graph $\underline{G \circ \mathbb{P}} := \langle (V', E'), \Sigma_0, \mathscr{C}' \rangle$ is defined in two steps:

1. **(Robustization:)** We convert each $c(e) \in \mathscr{C}$ to a circuit $\tilde{c}(e)$: For each $v \in V$, let $[v]$ be a set of ℓ Boolean variables. For each $e = (v, w) \in E$, $\tilde{c}(e)$ is a circuit with $2 \cdot \ell$ inputs $[v] \cup [w]$. $\tilde{c}(e)$ outputs 1 if and only if the assignment for $[v] \cup [w]$ is the legal encoding via e of an assignment for v and w that would have satisfied c.

Composition (cont'd)

2. **(Composition:)** Run the assignment tester $\mathbb{P}$ on each $\tilde{c}(e)$ which outputs $G_e = \langle (V_e, E_e), \Sigma_0, \mathscr{C}_e \rangle$. Assume, without loss of generality, E_e has the same cardinality for every $e \in E$. Finally, we define the constraint graph $G \circ \mathbb{P} := \langle (V', E'), \Sigma_0, \mathscr{C}' \rangle$ by

$$V' := \bigcup_{e \in E} V_e, \quad E' := \bigcup_{e \in E} E_e, \quad \mathscr{C}' := \bigcup_{e \in E} \mathscr{C}_e.$$

Composition Lemma

Lemma

Let $\mathbb{P}$ be an assignment tester with constant rejection probability $\varepsilon > 0$ and alphabet Σ_0. There exist a constant $\beta_3 > 0$ that depends only on $\mathbb{P}$, and a constant $c(\mathbb{P}, |\Sigma|)$ that depends only on $\mathbb{P}$ and $|\Sigma|$, such that the following holds.

Given any constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$, one can compute in time linear in $\mathrm{size}(G)$, the constraint graph $G' = G \circ \mathbb{P}$, with $\mathrm{size}(G') = c(\mathbb{P}, |\Sigma|) \cdot \mathrm{size}(G)$, and

$$\beta_3 \cdot \mathrm{unsat}(G) \leq \mathrm{unsat}(G') \leq \mathrm{unsat}(G).$$

Proof

$G' = G \circ \mathbb{P}$ can be computed in time linear in size(G)*:*

The first step (robustization) consists of $|E|$ steps of converting $c(e)$ to a circuit $\tilde{c}(e)$, with $2 \cdot \ell$ variables.
Thus, each conversion can clearly be done in time $2^{O(\ell)}$, which is a factor that depends ultimately only on $|\Sigma|$ and not on size(G).

In the second step, we feed each $\tilde{c}(e)$ to $\mathbb{P}$ to obtain the constraint graph G_e.

As the size of the input to $\mathbb{P}$ is bounded by some constant only depending on $|\Sigma|$, the running time of $\mathbb{P}$ is bounded by a constant, and the size of the output is also bounded by some constant.
Therefore, the blowup factor depends only on $|\Sigma|$ and on $\mathbb{P}$, i.e.,

$$\text{size}(G') = c(\mathbb{P}, |\Sigma|) \cdot \text{size}(G).$$

Proof (cont'd)

$\mathsf{unsat}(G') \leq \mathsf{unsat}(G)$*:*

Let $\sigma : V \to \Sigma$ be an assignment for G with

$$\mathsf{unsat}(G) = \mathsf{unsat}_\sigma(G).$$

We construct an assignment $\sigma' : V' \to \Sigma_0$ by *following the two steps in the definition of Composition*.

Recall that each vertex $v \in V$ is replaced by a set $[v]$ of vertices. Let

$$\sigma'([v]) = e(\sigma(v)) \in \{0,1\}^\ell$$

where $\sigma'([v])$ means the concatenation of $\sigma'(v)$ for all $y \in [v]$.

It remains to define values for σ' on

$$\bigcup_{e=(u,v)\in E} (V_e \setminus ([u] \cup [v]))$$

Proof (cont'd)

If for an $e = (u, v) \in E$ the constraint $c(e)$ is satisfied by σ, then the circuit $\tilde{c}(e)$ is satisfied by σ' restricted to $[u] \cup [v]$.

Then by the completeness property of $\mathbb{P}$, there is an extension assignment for $V_e \setminus ([u] \cup [v])$ that satisfies all constraints in G_e.

So if let a be the restriction of σ' to $[u] \cup [v]$, then there is some $b : V_e \setminus ([u] \cup [v]) \to \Sigma_0$ such that $\mathrm{unsat}_{a \cup b}(G_e) = 0$.

For the remaining vertices (belonging to graphs G_e whose constraint $c(e)$ is unsatisfied by σ) define σ' arbitrarily.
Since each E_e has the same cardinality, $\mathrm{unsat}_{\sigma'}(G') \leq \mathrm{unsat}_{\sigma}(G)$.
Therefore,

$$\mathrm{unsat}(G') \leq \mathrm{unsat}_{\sigma'}(G') \leq \mathrm{unsat}_{\sigma}(G) = \mathrm{unsat}(G).$$

Proof (cont'd)

$\beta_3 \cdot \mathrm{unsat}(G) \leq \mathrm{unsat}(G')$: Choose $\sigma' : V' \to \Sigma_0$ such that

$$\mathrm{unsat}_{\sigma'}(G') = \mathrm{unsat}(G')$$

We define $\sigma : V \to \Sigma$ by

$$\sigma(v) := \text{a value whose encoding via } e \text{ is } \textit{closest} \text{ to } \sigma'([v])$$

for every $v \in V$.

Let $F \subseteq E$ be the edges of G whose constraints are falsified by σ. Therefore,

$$\frac{|F|}{|E|} = \mathrm{unsat}_{\sigma}(G) \geq \mathrm{unsat}(G).$$

Let $e = (u, v) \in F$. We will show that *at least β_3 fraction of the constraints of G_e is falsified by σ'.*

Proof (cont'd)

Recall

$$G_e = \mathbb{P}(\tilde{c}(e)),$$

And $\sigma(u)$ and $\sigma(v)$ falsify $c(e)$ where $\sigma(u)$ ($\sigma(v)$, respectively) is the value whose encoding via e is *closest* to $\sigma'([u])$ ($\sigma'([v])$ respectively).

As e is an encoding of relative distance ρ, at least $\rho/2$ fraction of the bits in either $[u]$ or $[v]$ (or both) must be changed in order to change σ' into an assignment satisfying $\tilde{c}(e)$.

Thus, for every ξ that satisfies $\tilde{c}(e)$

$$\mathrm{rdist}\big(\sigma' \restriction ([u] \cup [v]), \xi\big) \geq \frac{\rho}{4}$$

Proof (cont'd)

By the soundness property of $\mathbb{P}$, at least $\varepsilon \cdot \rho/4$ fraction of the constraints in G_e are falsified.

Let $\beta_3 = \varepsilon \cdot \rho/4$.

$$\begin{aligned}
\mathsf{unsat}(G') &= \mathsf{unsat}_{\sigma'}(G') \\
&= \frac{|E|}{\sum_{e \in E} \mathsf{unsat}_{\sigma' \restriction V_e}(G_e)} \\
&\geq \frac{|E|}{\sum_{e \in F} \mathsf{unsat}_{\sigma' \restriction V_e}(G_e)} \\
&\geq \frac{\beta_3 \cdot |F|}{|E|} = \beta_3 \cdot \mathsf{unsat}_{\sigma}(G) \geq \beta_3 \cdot \mathsf{unsat}(G).
\end{aligned}$$

□