# Expander Graphs and Their Applications (I)

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# Two Motivating Examples

# Graph Reachability Problem

| | |
|---|---|
| *Input:* | A graph $G = (V, E)$ and $u, v \in V$. |
| *Problem:* | Is there a path in $G$ from $u$ to $v$? |

Recall Breadth-first Search can solve the problem in time $O(|V| + |E|)$.

BFS seems perfect . . . .

What if $G$ is the World Wide Web, i.e., a huge graph.

- Then we cannot store the graph in our computer!

  Not a problem, since we can treat WWW as "stored in the *external* memory."

- We cannot store all the intermediate results in our computer.

  A BIG problem, since we can only read the external memory, not *write* it.

# Graph Reachability Problem on Good Graphs

For our purpose, let $c, d \in \mathbb{N}$, a graph $G = (V, E)$ is $(c, d)$-good if each $v \in V$ has degree at most $d$ and for all $u, v \in V$, if there is a path from $u$ to $v$, then

$$\text{dist}(u, v) \leq c \cdot \log |V|.$$

A *space-efficient* algorithm:

$\textsc{Distance}_d(G, u, v, \ell)$
// $G = (V, E)$ a graph, $u, v \in V$, and $\ell \in \mathbb{N}$

1. **if** $u = v$ **then** return '1.'
2. **if** $\ell = 0$ **then** return '0.'
3. **for** $i = 1$ **to** $d$ **do**
4. $\quad w \leftarrow$ *the i-th neighbor* of $u$.
5. $\quad$ **if** $\textsc{Distance}_d(G, w, v, \ell - 1)$ returns '1' **then** return '1.'
6. return '0.'

# Graph Reachability Problem on Good Graphs (cont'd)

**Theorem**

*Let $G = (V, E)$ be a graph such that every vertex has degree at most $d$. Then for every $u, v \in V$ and $\ell \in \mathbb{N}$.*

$$\textsc{Distance}_d(G, u, v, \ell) = 1 \iff \text{dist}(u, v) \leq \ell.$$

**Theorem**

$\textsc{Distance}_d(G, u, v, \ell)$ *uses memory/space:* $O(\ell \cdot \log d)$ *bits.*

**Corollary**

*Let $c, d \in \mathbb{N}$. For $(c, d)$-good graphs $G = (V, E)$, the reachability problem can be solved in space*

$$O(c \cdot \log |V| \cdot \log d) = O(\log |V|),$$

*i.e., in* $\textsc{LogSpace}$.

# Graph Reachability in LogSpace

Theorem (**Omer Reingold**, 2005)

*There is an algorithm that solves the reachability problem on any graph* $G = (V, E)$ *using space* $O(\log |V|)$.

Idea: *Reduce* the general problem to "good graphs," even stronger: Expander Graphs.

# Approximation Algorithms for MAX-INDEPENDENTSET?

Recall:

MAX-INDEPENDENTSET

| | |
|---|---|
| *Input:* | A graph $G = (V, E)$. |
| *Solution:* | An independent set $I \subseteq V$. |
| *Cost:* | $\|I\|$. |
| *Goal:* | max. |

MIN-VERTEXCOVER

| | |
|---|---|
| *Input:* | A graph $G = (V, E)$. |
| *Solution:* | A vertex cover $C \subseteq V$. |
| *Cost:* | $\|C\|$. |
| *Goal:* | min. |

**Theorem**

*There is a polynomial time 2-approximation for* MIN-VERTEXCOVER.

# Approximation Algorithms for Max-IndependentSet (cont'd)?

**Theorem (Arora, et.al, 1992)**

*There is no 2-approximation for* Max-IndependentSet, *unless* **NP=P**.

The major tool is the PCP Theorem (**Arora**, et.al, 1992], i.e., probabilistic checkable proof.

The original proof of PCP Theorem is algebraic (but somehow elementary), with $\approx$ 100 pages.

In 2006, **Irit Dinur** gave a combinatorial proof for the PCP Theorem, non-elementary, with $\approx$ 40 pages.

The major tool of Dinur's proof is Expander Graphs.

# Introduction to Expander Graphs

# Graph Expansion

A (undirected) graph $G = (V, E)$ can have selfloops and multi-edges.

For every $e \in E$ with end-vertices $u$ and $v$, it has two directions: $(e, u, v)$ and $(e, v, u)$. If $e$ is a selfloop, i.e., $u = v$, then its two directions are the same. For a direction $(e, u, v)$, $u$ is its tail and $v$ its head.

For every $S, T \subseteq V$ we let

$$E(S, T) := \left\{ \vec{e} \mid \vec{e} \text{ is a direction of some } e \in E \text{ with tail in } S \text{ and head in } T \right\}.$$

Then $e(S, T) := |E(S, T)|$. Note every edge can be *counted twice* except for those selfloops. Therefore it might not hold that $e(V, V) = 2|E|$, but

$$e(V, V) = d|V|$$

in case $G$ is $d$-regular.

# Graph Expansion (cont'd)

Let $S \subseteq V$. Then the edge boundary of $S$ is

$$\partial S := E(S, \bar{S})$$

where $\bar{S} := V \setminus S$.

The (edge) expansion ratio of $G$ is

$$h(G) := \min_{\{S \mid |S| \leq |V|/2\}} \frac{|\partial S|}{|S|}.$$

- If $G$ is a complete graph (i.e., a clique), then $h(G) = \lceil |V|/2 \rceil$.
- If every vertex in $G$ has degree $\leq d$, then $h(G) \leq d$.
- If $G$ is not connected, then $h(G) = 0$.
- If $G$ is a full binary tree, then $h(G) = 2/(|V| - 1)$.

# Definition of Expander Graphs

## Definition

Let $d \in \mathbb{N}$. A sequence of *d-regular graphs* $\{G_i\}_{i \in \mathbb{N}}$ of *size increasing with i* is a family of expander graphs if there exists $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$ for all $i$.

All the previous examples are not family of expander graphs.

# Graph spectrum and an algebraic definition of expansion

# Graphs as Matrices

The adjacency matrix of an $n$-vertex graph $G$, denoted $A = A(G)$, is an $n \times n$ matrix whose *$(u, v)$ entry is the number of edges in $G$ between vertex $u$ and vertex $v$*. That is, $A = (a_{i,j})_{i,j \in [n]}$ where

$$a_{i,j} = \text{the number of edges in } G \text{ between the } i\text{-th and the } j\text{-th vertices.}$$

For every graph $G$, $A(G)$ is *real* and *symmetric*.

If $G$ is $d$-regular and $A = A(G) = (a_{i,j})_{i,j \in [n]}$, then for every $i \in [n]$

$$\sum_{k \in [n]} a_{i,k} = d = \sum_{k \in [n]} a_{k,i}.$$

# Some basic linear algebra

Let $x, y \in \mathbb{R}^n$, i.e., $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then the inner product (or dot product) of $x$ and $y$ is

$$\langle x, y \rangle = \sum_{i \in [n]} x_i y_i.$$

$x$ and $y$ are orthogonal (or perpendicular) if

$$\langle x, y \rangle = 0.$$

The "length" of $x$ is

$$\|x\|_2 = \sqrt{\langle x, x \rangle}.$$

(That is, the $\ell_2$ norm of $x$.)

The normalization of $x$ is

$$\frac{x}{\|x\|_2},$$

i.e, the vector with the same direction as $x$ yet of length 1.

# Some basic linear algebra (cont'd)

Let $A$ be an $n \times n$ real matrix. If there is a non-zero vector $x \in \mathbb{R}^n$ such that

$$Ax = \lambda x$$

for some $\lambda \in \mathbb{R}$. Then $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $x$.

### Theorem

*If $A$ is real and symmetric, then*

- *$A$ has exactly $n$ (not necessarily distinct) eigenvalues, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.*
- *There exists a set of $n$ eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, one for each eigenvalue (i.e., $Av_i = \lambda_i v_i$), that are orthonormal, i.e., all $v_i$s are of length 1 and orthogonal with each other.*
  *Note, $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$.*

# Graph Spectrum

Let $G$ be a graph and $A = A(G)$. The spectrum of $G$ is the eigenvalues of $A$, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Let $G$ be a $d$-regular graph.

- $\lambda_1 = d$ and the corresponding eigenvectors is

$$v_1 = \frac{\mathbf{1}}{\sqrt{n}} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right).$$

- The graph is connected if and only if $\lambda_1 > \lambda 2$.
- The graph is bipartite if and only if $\lambda_1 = -\lambda_n$.

# Proofs

Let $\lambda$ be an eigenvalue of $A = A(G) = (a_{i,j})_{i,j\in[n]}$ with corresponding eigenvector $v = (e_1, \ldots, e_n)$.
We choose an $i \in [n]$ such that for every $j \in [n]$

$$|e_i| \geq |e_j|.$$

By $Av = \lambda v$,

$$\sum_{k\in[n]} a_{i,k} e_k = \lambda e_i,$$

Then,

$$d|e_i| = |e_i| \sum_{k\in[n]} |a_{i,k}| \geq \sum_{k\in[n]} |a_{i,k}||e_k| \geq \left| \sum_{k\in[n]} a_{i,k} e_k \right| = |\lambda||e_i|.$$

Thus, $\lambda_1 \leq d$. ⊣

**Remark.** *We actually proved that*

$$d = \lambda_1 \geq \ldots \geq \lambda_n \geq -d.$$

# Expansion as Spectrum Gap

**Theorem**

*Let $G$ be a $d$-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then*

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

*Note $d - \lambda_2 = \lambda_1 - \lambda_2$ is the spectral gap of $G$.*

# Expander Mixing Lemma

We let $\underline{\lambda} := \underline{\lambda(G)} := \max\{|\lambda_2|, |\lambda_n|\}$.

Lemma

*Let $G = (V, E)$ be a $d$-regular graph with $n$-vertices. Then for all $S, T \subseteq V$,*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.$$

# $\lambda$ and $\lambda_2$

Recall: $\lambda = \max\left\{|\lambda_2|, |\lambda_n|\right\}$. Thus, it is necessarily true that $\lambda = \lambda_2$.

Let $G = (V, E)$ be a graph and $A = A(G)$. Then $A^2$ is the adjacency matrix of $G^2 = (V', E')$ where $V' := V$ and for every $u, v \in V$

$$\begin{aligned} & \text{the number of edges between } u \text{ and } v \text{ in } E' \\ = \; & \text{the number of length-2 walks between } u \text{ and } v \text{ in } G. \end{aligned}$$

Let $\lambda_1, \ldots, \lambda_n$ be the spectrum of $G$ with corresponding eigenvectors $v_1, \ldots, v_n$. Then $\lambda_1^2, \ldots, \lambda_n^2$ are the spectrum of $G^2$ with corresponding eigenvectors $v_1, \ldots, v_n$.

It follows $\lambda(G^2) = \lambda_2(G^2)$.

# Proof of the Expander Mixing Lemma

Let $A = A(G)$ with corresponding eigenvalues $d = \lambda_1 \geq \ldots \geq \lambda_n$. We fix a corresponding *orthonormal* set of eigenvectors $v_1, \ldots, v_n$ with

$$v_1 = \frac{\mathbf{1}}{\sqrt{n}} = \left( \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right).$$

Let $1_S$ and $1_T$ be the characteristic vectors for $S$ and $T$. Assume

$$1_S = \sum_{i \in [n]} \alpha_i v_i \quad \text{and} \quad 1_T = \sum_{i \in [n]} \beta_i v_i$$

Then

$$\left|E(S,T)\right| = 1_S A 1_T = \left( \sum_{i \in [n]} \alpha_i v_i \right) A \left( \sum_{i \in [n]} \beta_i v_i \right) = \sum_{i,j \in [n]} (\alpha_i v_i) A (\beta_j v_j)$$
$$= \sum_{i,j \in [n]} \lambda_j \alpha_i \beta_j \langle v_i, v_j \rangle = \sum_{i \in [n]} \lambda_i \alpha_i \beta_i \langle v_i, v_i \rangle = \sum_{i \in [n]} \lambda_i \alpha_i \beta_i.$$

# Proof (cont'd)

Note

$$\alpha_1 = \left\langle 1_S, \frac{\mathbf{1}}{\sqrt{n}} \right\rangle = \frac{|S|}{\sqrt{n}}, \qquad \beta_1 = \left\langle 1_T, \frac{\mathbf{1}}{\sqrt{n}} \right\rangle = \frac{|T|}{\sqrt{n}}, \quad \text{and } \lambda_1 = d.$$

Then

$$\left|E(S,T)\right| = \sum_{i \in [n]} \lambda_i \alpha_i \beta_i = \frac{d|S||T|}{n} + \sum_{2 \le i \le n} \lambda \alpha_i \beta_i$$

$$\left| \left|E(S,T)\right| - \frac{d|S||T|}{n} \right| = \left| \sum_{2 \le i \le n} \lambda_i \alpha_i \beta_i \right| \le \sum_{2 \le i \le n} \left|\lambda_i \alpha_i \beta_i\right| \le \lambda \sum_{2 \le i \le n} \left|\alpha_i \beta_i\right|$$

Finally by **Cauchy-Schwartz**:

$$\left| \left|E(S,T)\right| - \frac{d|S||T|}{n} \right| \le \lambda \|\alpha\|_2 \|\beta\|_2 = \lambda \|1_S\|_2 \|1_T\|_2 = \lambda \sqrt{|S||T|}.$$

□