Expander Graphs and Their Applications (XX)

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Constraints

Definition

Let $V = \{v_1, \dots, v_n\}$ be a set of variables, and let Σ be a finite alphabet. A <u>q-ary constraint</u> $c = (C, i_1, \dots, i_q)$ consists of

- a q-tuple of indices $i_1, \dots, i_q \in [n]$,
- and a subset $C \subseteq \Sigma^q$ of "acceptable" values.

An assignment $\mathcal{A} : V \to \Sigma$ *satisfies* c if

$$(\mathcal{A}(v_{i_1}), \dots, \mathcal{A}(v_{i_q})) \in C.$$

The formula $X_3 \vee \neg X_7 \vee X_9$ can be viewed as a constraint $(C, 3, 7, 9)$ with

$$\{(1,0,0),(1,0,1),(1,1,0),(1,1,1),(0,0,0),(0,0,1),(0,1,1)\}.$$

Constraint Satisfaction Problems (CSP)

CSP

Input:	A system (i.e., set) $\mathscr{C}$ of constraints over a set of variables V and a finite alphabet Σ.
Problem:	Is there an assignment $\mathcal{A} : V \to \Sigma$ that satisfies every constraint?

3COLORABILITY as CSP

Lemma

3COLORABILITY *is reducible to* CSP.

Proof.

Let $\Sigma := \{0, 1, 2\}$. Given a graph $G = (V, E)$, we identify V with the set of variables.
For each edge $e \in E$ with end vertices v_i and v_j, we introduce a constraint

$$c_e := (C_e, i, j)$$

where

$$C_e := \{(a, b) \in \Sigma^2 \mid a \neq b\}.$$

Then

$$G \text{ is 3-colorable} \iff \text{there is an assignment satisfying } \{C_e \mid e \in E\}.$$

□

unsat

Let $\mathscr{C}$ be a set of constraints. We define

$$\begin{aligned}\underline{\text{unsat}(\mathscr{C})} := &\text{ the smallest fraction of}\\ &\qquad \textit{unsatisfied} \text{ constraints of } \alpha \text{ under any assignment}\\ = &\min_{\mathcal{A} \text{ an assignment}} \frac{\left|\{c \in \mathscr{C} \mid \mathcal{A} \text{ satisfies } c\}\right|}{|\mathscr{C}|}.\end{aligned}$$

$$\begin{aligned}\mathscr{C} \text{ is satisfiable} \quad &\Longleftrightarrow \quad \text{unsat}(\mathscr{C}) = 0;\\ \mathscr{C} \text{ is not satisfiable} \quad &\Longleftrightarrow \quad \text{unsat}(\mathscr{C}) \geq \frac{1}{|\mathscr{C}|}.\end{aligned}$$

Gap-CSP

Let $q \in \mathbb{N}$, $\varepsilon > 0$, and Σ be a finite alphabet.

$\textsc{Gap-CSP}_{q,\Sigma,\epsilon}$	
Input:	A set $\mathscr{C}$ of q-ary constraints over the alphabet Σ such that *either* $\text{unsat}(\mathscr{C}) = 0$ *or* $\text{unsat}(\mathscr{C}) \geq \epsilon$.
Problem:	Decide whether $\mathscr{C}$ is satisfiable.

Theorem

$\text{NP} = \text{PCP}(\log n, 1)$ *if and only if for some* $\varepsilon > 0$ *the problem* $\textsc{Gap-CSP}_{q,\Sigma,\epsilon}$ *is* NP-*hard (for some* $q \geq 2$ *and* $|\Sigma| \geq 2$*).*

Constraint Graphs and Operations on Them

Constraint Graph

Definition

$G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ is a constraint graph, if

1. (V, E) is an undirected graph (possibly with selfloops and multi-edges), i.e., *the underlying graph* of G.
2. The set V is also viewed as a set of variables assuming values over alphabet Σ.
3. Each edge $e \in E$ carries a constraint $c(e) \subseteq \Sigma^2$ and $\mathscr{C} = \{c(e) \mid e \in E\}$. A constraint $c(e)$ is said to be satisfied by (a, b) if $(a, b) \in c(e)$.

Remark.

- *The above definition is the rephrase of a CSP with each constraint being binary.*
- *Sometimes, we also use G to refer to (V, E).*

unsat

An assignment is a mapping $\sigma : V \to \Sigma$.

For a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ and an assignment σ:

$$\underline{\text{unsat}_\sigma(G)} := \Pr_{\substack{e \in E \text{ with} \\ \text{endvertices } u \text{ and } v}} \left[(\sigma(u), \sigma(v)) \notin c(e) \right]$$

Then

$$\underline{\text{unsat}(G)} := \min_\sigma \text{unsat}_\sigma(G)$$

We have already seen:

Theorem

Given a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ with $|\Sigma| = 3$, it is NP*-hard to decide whether* $\text{unsat}(G) = 0$.

unsat Amplification

Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph. Then

$$G \text{ is satisfiable} \iff \text{unsat}(G) = 0$$
$$G \text{ is not satisfiable} \iff \text{unsat}(G) \geq \frac{1}{|\mathscr{C}|} = \frac{1}{|E|}.$$

Our goal: In polynomial time, compute from G another constraint graph G^* such that

$$G \text{ is satisfiable} \iff \text{unsat}(G^*) = 0$$
$$G \text{ is not satisfiable} \iff \text{unsat}(G^*) \geq \alpha.$$

Main Theorem

Definition

For every constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$, let

$$\text{size}(G) := |V| + |E|.$$

Theorem

There exists a Σ_0 such that the following hold.

For every finite alphabet Σ there exist two constants $C_\Sigma > 0$ and $0 < \alpha_\Sigma < 1$ and a polynomial time algorithm $\mathbb{A}_\Sigma$ such that for every constraint graph G, the algorithm $\mathbb{A}_\Sigma$ computes a constraint graph G' such that

- $\text{size}(G') \leq C_\Sigma \cdot \text{size}(G)$.
- *If* $\text{unsat}(G) = 0$, *then* $\text{unsat}(G') = 0$.
- *If* $\text{unsat}(G) \neq 0$, *then* $\text{unsat}(G') \geq \min(2 \cdot \text{unsat}(G), \alpha_\Sigma)$.

Main Theorem (cont'd)

Let Σ be a finite alphabet.

Consider the following algorithm $\mathbb{G}_\Sigma$ with input a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$.

1. $G_0 = \langle (V_0, E_0), \Sigma_0, \mathscr{C}_0 \rangle \leftarrow \mathbb{A}_\Sigma(G)$.
2. **for** $i = 1$ **to** $\log |\mathscr{C}_0|$ **do**
3. $\quad G_i \leftarrow \mathbb{A}_{\Sigma_0}(G)$.
4. Output $G_{\log |\mathscr{C}_0|}$.

Theorem

- $\text{unsat}(\mathbb{G}_\Sigma(G)) \geq \alpha_{\Sigma_0}$.
- *For every* $i \in [\log |\mathscr{C}_0|]$,

$$\text{size}(G_i) \leq C_{\Sigma_0}^{i} \cdot \text{size}(G_0) = C_{\Sigma_0}^{\log |\mathscr{C}_0|} \cdot \text{size}(G_0) = \text{size}(G)^{O(1)}.$$

- $\mathbb{G}_\Sigma$ *is a polynomial time algorithm.*

Graph Powering

Fix some $d \in \mathbb{N}$.

Let $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ be a constraint graph with (V, E) *being d-regular*, and $t \in \mathbb{N}$.

Recall, a sequence $(u_0, u_1, \ldots, u_t)$ is a t-step walk in G if there is an edge between u_{i-1} and u_i for all $i \in [t]$.

Then we will define the following t-th power of G

$$\underline{G^t} := \left\langle (V, E^t), \Sigma^{d^{\lceil t/2 \rceil}}, \mathscr{C}^t \right\rangle$$

$G^t := \left\langle (V, E^t), \Sigma^{d^{\lceil t/2 \rceil}}, \mathscr{C}^t \right\rangle$

1. The vertices of G^t are the same as G. The number of edges in E^t between u and v is the number of t-step walks from u to v in G.
2. The alphabet of G^t is $\Sigma^{d^{\lceil t/2 \rceil}}$: Let

   $$\underline{\Gamma(u)} := \left\{ u' \in V \mid u = u_0, u_1, \ldots, u_{\lceil t/2 \rceil} = u' \text{ is a walk in } G \right\}.$$

   Then $|\Gamma(u)| \leq d^{\lceil t/2 \rceil}$ and *by choosing some canonical order*, a value $a \in \Sigma^{d^{\lceil t/2 \rceil}}$ can be interpreted as an assignment $a : \Gamma(u) \to \Sigma$. One might think of this value as describing u's opinion of its neighbor's values.
3. The constraint $C(e)$ associated with an edge $e \in E^t$ with end vertices u and v contains those pairs $a, b \in \Sigma^{d^{\lceil t/2 \rceil}}$ if: There is an assignment $\sigma : \Gamma(u) \cup \Gamma(v) \to \Sigma$ that satisfies every constraint $c(e) \in \mathscr{C}$ where $e \in E \cap \big(\Gamma(u) \times \Gamma(v)\big)$, and such that

   $$\text{for all } u' \in \Gamma(u) \text{ and } v' \in \Gamma(v), \quad \sigma(u') = a_{u'} \text{ and } \sigma(v') = b_{v'}$$

   where $a_{u'}$ is the value a assigns u', and $b_{v'}$ the value b assigns to v'.

Amplification Lemma

Clearly $\text{unsat}(G) = 0$ implies $\text{unsat}(G^t) = 0$. What about the other direction?

Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d-regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$,

$$\text{unsat}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}(G), \frac{1}{t}\right).$$

What we have lost:

- degree $d \to d^t$;
- size $|V| + |E| \to |V| + d^{t-1} \cdot |E|$.

Preprocessing Lemma

Lemma

There exist constant $0 < \lambda < d$ *and* $\beta_1 > 0$ *such that any constraint graph* G *can be transferred into a constraint graph* $G' := \underline{\text{prep}(G)}$ *such that*

- G' *is* d*-regular with selfloops and* $\lambda(G') \leq \lambda < d$.
- G' *has the same alphabet as* G *and* $\text{size}(G') = O(\text{size}(G))$.
- $\beta_1 \cdot \text{unsat}(G) \leq \text{unsat}(G') \leq \text{unsat}(G)$.

Alphabet Reduction

Lemma

There exist a constant $\beta_3 > 0$, an alphabet Σ_0, and a linear time algorithm $\mathbb{C}$ such that for every constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ the algorithm $\mathbb{C}$ computes a constraint graph G' over the alphabet Σ_0 such that

- $\text{size}(G') \leq c_\Sigma \cdot \text{size}(G)$, *where c_Σ is a constant only depending on Σ.*
- *and* $\beta_3 \cdot \text{unsat}(G) \leq \text{unsat}(G') \leq \text{unsat}(G)$.

Main Theorem (again)

Theorem

There exists a Σ_0 *such that the following hold.*

For every finite alphabet Σ *there exist two constants* $C_\Sigma > 0$ *and* $0 < \alpha_\Sigma < 1$ *and a polynomial time algorithm* $\mathbb{A}_\Sigma$ *such that for every constraint graph* G*, the algorithm* $\mathbb{A}_\Sigma$ *computes a constraint graph* G' *such that*

- $\mathrm{size}(G') \leq C_\Sigma \cdot \mathrm{size}(G)$.
- *If* $\mathrm{unsat}(G) = 0$*, then* $\mathrm{unsat}(G') = 0$.
- *If* $\mathrm{unsat}(G) \neq 0$*, then* $\mathrm{unsat}(G') \geq \min(2 \cdot \mathrm{unsat}(G), \alpha_\Sigma)$.

Proof of the Main Theorem

Let

$$
\begin{aligned}
H_1 &:= \mathrm{prep}(G) \\
H_2 &:= (H_1)^t \\
G' &:= H_2 \longrightarrow G'
\end{aligned}
$$

where

$$t := \left\lceil \left(\frac{2}{\beta_1 \cdot \beta_2 \cdot \beta_3} \right)^2 \right\rceil$$

and the arrow is the the alphabet reduction.

Proof of the Main Theorem (cont'd)

Then

$$\begin{aligned}
\mathrm{unsat}(G') &\geq \beta_3 \cdot \mathrm{unsat}(H_2) \\
&\geq \beta_3 \cdot \beta_2 \cdot \sqrt{t} \cdot \min\left(\mathrm{unsat}(H_1), \frac{1}{t}\right) \\
&\geq \beta_3 \cdot \beta_2 \cdot \sqrt{t} \cdot \min\left(\beta_1 \cdot \mathrm{unsat}(G), \frac{1}{t}\right) \\
&\geq \min\left(2 \cdot \mathrm{unsat}(G), \alpha\right)
\end{aligned}$$

where

$$\alpha := \frac{\beta_3 \cdot \beta_2}{\sqrt{t}}.$$

□