Expander Graphs and Their Applications (II)

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Last Lecture

Graph Reachability Problem on Good Graphs

For our purpose, let $c, d \in \mathbb{N}$, a graph $G = (V, E)$ is (c, d)-good if each $v \in V$ has degree at most d and for all $u, v \in V$, if there is a path from u to v, then

$$\mathrm{dist}(u, v) \leq c \cdot \log |V|.$$

Theorem

Let $c, d \in \mathbb{N}$. For (c, d)-good graphs $G = (V, E)$, the reachability problem can be solved in space

$$O(c \cdot \log |V| \cdot \log d) = O(\log |V|),$$

i.e., in LogSpace.

Graph Expansion

For every $S, T \subseteq V$ we let

$E(S,T) := \{\vec{e} \mid \vec{e}$ is a direction of some $e \in E$ with tail in S and head in $T\}$.

Let $S \subseteq V$. Then the edge boundary of S is

$$\partial S := E(S, \bar{S}).$$

The (edge) expansion ratio of G is

$$h(G) := \min_{\{S \mid |S| \leq |V|/2\}} \frac{|\partial S|}{|S|}.$$

Definition

Let $d \in \mathbb{N}$. A sequence of *d-regular graphs* $\{G_i\}_{i \in \mathbb{N}}$ of *size increasing with i* is a family of expander graphs if there exists $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$ for all i.

Graphs as Matrices

The adjacency matrix of an n-vertex graph G, denoted $A = A(G)$, is an $n \times n$ matrix $(a_{i,j})_{i,j \in [n]}$ where

$$a_{i,j} = \text{the number of edges in } G \text{ between the } i\text{-th and the } j\text{-th vertices.}$$

Note, A is a real and symmetric matrix.

Theorem

If A is real and symmetric, then

- *A has exactly n (not necessarily distinct) eigenvalues, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.*
- *There exists a set of n eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, one for each eigenvalue (i.e., $Av_i = \lambda_i v_i$), that are orthonormal, i.e., all v_is are of length 1 and orthogonal with each other.*

Note, $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$.

Graph Spectrum

Let G be a graph and $A = A(G)$. The spectrum of G is the eigenvalues of A, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Let G be a d-regular graph. Then $\lambda_1 = d$ and the corresponding eigenvector can be chosen as

$$v_1 = \frac{\mathbf{1}}{\sqrt{n}} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right).$$

Theorem

Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

Note $d - \lambda_2 = \lambda_1 - \lambda_2$ is the spectral gap of G.

Expander Mixing Lemma

We let $\underline{\lambda} := \underline{\lambda(G)} := \max\{|\lambda_2|, |\lambda_n|\}$.

Lemma

Let $G = (V, E)$ be a d-regular graph with n-vertices. Then for all $S, T \subseteq V$,

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda\sqrt{|S||T|}.$$

d-Regular Graphs as Matrices

Lemma

$\lambda_1 = d$ and the corresponding eigenvector is $v_1 = \mathbf{1}/\sqrt{n}$.

Proof.

Let λ be any eigenvalue of $A = A(G) = (a_{i,j})_{i,j\in[n]}$ with corresponding eigenvector $v = (e_1, \ldots, e_n)$. We choose an $i \in [n]$ such that for every $j \in [n]$

$$|e_i| \geq |e_j|.$$

By $Av = \lambda v$,

$$\sum_{k\in[n]} a_{i,k} e_k = \lambda e_i,$$

Then,

$$d|e_i| = |e_i| \sum_{k\in[n]} |a_{i,k}| \geq \sum_{k\in[n]} |a_{i,k}||e_k| \geq \left| \sum_{k\in[n]} a_{i,k} e_k \right| = |\lambda||e_i|.$$

$\square$

d-Regular Graphs as Matrices (cont'd)

Lemma

The graph is connected if and only if $\lambda_1 > \lambda 2$.

Proof.

($\Leftarrow$) Easy.

($\Rightarrow$) Let $v = (e_1, \ldots, e_n)$ be an eigenvector for d. We choose an $i \in [n]$ such that for every $j \in [n]$

$$|e_i| \geq |e_j|.$$

Without loss of generality, we assume $e_i > 0$. Then,

$$de_i = e_i \sum_{k \in [n]} a_{i,k} = e_i \sum_{\{i,k\} \in E} a_{i,k} \geq \sum_{\{i,k\} \in E} a_{i,k} e_k = \sum_{k \in [n]} a_{i,k} e_k = de_i,$$

where $\{i, k\} \in E$ means that *there is at least one edge between the i-th and the k-th vertices*.

Therefore, if $\{i, k\} \in E$, then $e_k = e_i$. Then we repeat the above argument on e_k.

□

d-Regular Graphs as Matrices (cont'd)

Lemma

The graph is bipartite if and only if $\lambda_1 = -\lambda_n$.

Proof.

($\Rightarrow$) Easy.

($\Leftarrow$) Let $v = (e_1, \ldots, e_n)$ be an eigenvector for $-d$. We choose an $i \in [n]$ such that for every $j \in [n]$

$$|e_i| \geq |e_j|.$$

Then,

$$de_i = e_i \sum_{k \in [n]} a_{i,k} = e_i \sum_{\{i,k\} \in E} a_{i,k}$$

On the other hand,

$$de_i = -(-de_i) = -\sum_{k \in [n]} a_{i,k} e_k = \sum_{\{i,k\} \in E} a_{i,k}(-e_k).$$

Therefore, if $\{i, k\} \in E$, then $e_k = -e_i$. Then we repeat the above argument on e_k.

□

Expander Graphs are Almost Good

Recall: A graph $G = (V, E)$ is (c, d)-good if each $v \in V$ has degree at most d and for all $u, v \in V$, if there is a path from u to v, then

$$\text{dist}(u, v) \leq c \cdot \log |V|.$$

Definition

A d-regular graph G on n vertices is called an $\underline{(n, d)\text{-graph}}$. Moreover, it is an $\underline{(n, d, \alpha)\text{-graph}}$ if $\lambda(G) \leq \alpha d$.

Theorem

Let $\alpha < 1/2$. Then all (n, d, α) graphs are (c, d)-good for some appropriate $c \in \mathbb{N}$.

Proof

Fix an (n, d, α)- graph $G = (V, E)$. For every $v \in V$ and $r \in \mathbb{N}$ let

$$\underline{B(x, r)} := \{y \in V \mid \mathrm{dist}(x, y) \leq r\}.$$

Claim. If $|B(x, r)| \leq n/2$, then for some fixed $\varepsilon > 0$

$$\big|B(x, r+1)\big| \geq (1+\varepsilon)\big|B(x, r)\big|.$$

By the Expander Mixing Lemma, for every $S \subseteq V$

$$\frac{|E(S, S)|}{|S|} \leq d\left(\frac{|S|}{n} + \alpha\right).$$

Then by $|E(S, \bar{S})| + |E(S, S)| = d|S|$,

$$\frac{|E(S, \bar{S})|}{|S|} \geq d\left((1-\alpha) - \frac{|S|}{n}\right).$$

S has at least $|E(S, \bar{S})|/d$ neighbors *outside* S. The claim follows by taking $\varepsilon := 1/2 - \alpha$. ⊣

Then for some $r = O(\log n)$ and every $x \in V$ we have $|B(x, r)| > n/2$. □

Random Walks on Expander Graphs

Random Walks

A vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ is a probability (distribution) vector if all $p_i \geq 0$ and $\sum_{i \in [n]} p_i = 1$. The probability vector for the *uniform distribution* is

$$\underline{u} := \frac{\mathbf{1}}{n} = \left\langle \frac{1}{n}, \ldots, \frac{1}{n} \right\rangle.$$

Definition

A random walk on a graph $G = (V, E)$ is a *discrete-time stochastic process* $(X_0, X_1, \ldots)$ taking values in V. The vertex X_0 is sampled from some *initial distribution* on V, and X_{i+1} is chosen *uniformly at random from the neighbors of X_i*.

Normalized Adjacency Matrices

Let G be a d-regular graph with adjacency matrix $A = A(G)$. Then its normalized adjacency matrix is

$$\hat{A} := \frac{1}{d} A.$$

- The random walk on $G = (V, E)$ is a Markov Chain with state set V and transition matrix $\hat{A}$.
- $\hat{A}$ s real, symmetric, and doubly stochastic; i.e. every column and every row sums up to 1.
- If $d = \lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of A, then $1 = \hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_n$ with $\hat{\lambda}_i = \lambda_i / d$ for $i \in [n]$ are the eigenvalues of $\hat{A}$ with the same eigenvectors as of A.
- Sampling a vertex x from some probability distribution p on V and then moving to a random neighbor of x is equivalent to sampling a vertex from the distribution $\hat{A}p$.

Normalized Adjacency Matrices (cont'd)

- The matrix $\hat{A}^t$ is the transition matrix of the Markov Chain defined by *random walks of length t*, i.e., $(\hat{A}^t)_{i,j}$ is the probability *a random walk starting at i is at j after t steps*.
- The *stationary distribution* of the random walk on G is the uniform distribution, namely, $u\hat{A} = \hat{A}u = u$.

The norms of vectors

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then:

- $\|x\|_1 := \sum_{i \in [n]} |x_i|$.
- $\|x\|_2 := \sqrt{\sum_{i \in [n]} x_i^2}$.
- $\|x\|_\infty := \max_{i \in [n]} |x_i|$.

Rapid Mixing of Walks

Theorem

Let G be an (n, d, α)-graph with normalized adjacency matrix $\hat{A}$. Then for any distribution vector p and any positive integer t:

$$\|\hat{A}^t p - u\|_1 \leq \sqrt{n}\,\alpha^t$$

Corollary

(n, d, α)-graphs are (c, d)-good with

$$c = -\frac{3}{2\log\alpha} + \varepsilon.$$

and $\varepsilon > 0$.

Rapid Mixing of Walks (cont'd)

Theorem

Let G be an (n, d, α)-graph with normalized adjacency matrix $\hat{A}$. Then for any distribution vector p and any positive integer t:

$$\|\hat{A}^t p - u\|_2 \leq \|p - u\|_2 \alpha^t \leq \alpha^t.$$

Lemma

For every probability vector p, $\|\hat{A}p - u\|_2 \leq \|p - u\|_2 \alpha \leq \alpha$.

Proof of the Lemma

Obviously, $\langle p - u, u \rangle = 0$, that is, $p - u$ and u are orthogonal. It is easy to show $\|p - u\|_2 \leq 1$. Consequently, for some appropriate $(\alpha_i)_{2 \leq i \leq n}$ we have

$$p - u = \sum_{2 \leq i \leq n} \beta_i v_i .$$

$$\begin{aligned}
\|\hat{A}p - u\|_2 &= \|\hat{A}p - \hat{A}u\|_2 = \|\hat{A}(p - u)\|_2 \\
&= \left\| \sum_{2 \leq i \leq n} \hat{\lambda}_i \beta_i v_i \right\|_2 = \sqrt{\left\langle \sum_{2 \leq i \leq n} \hat{\lambda}_i \beta_i v_i , \sum_{2 \leq i \leq n} \hat{\lambda}_i \beta_i v_i \right\rangle} \\
&= \sqrt{\sum_{2 \leq i \leq n} \hat{\lambda}_i^2 \beta_i^2 \langle v_i, v_i \rangle} \leq \alpha \sqrt{\sum_{2 \leq i \leq n} \beta_i^2 \langle v_i, v_i \rangle} \\
&= \alpha \left\| \sum_{2 \leq i \leq n} \beta_i v_i \right\|_2 = \alpha \|p - u\|_2 \leq \alpha .
\end{aligned}$$

□