Expander Graphs and Their Applications (III)

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Last Lecture

Expander Graphs are Almost Good

Recall: A graph $G = (V, E)$ is (c, d)-good if each $v \in V$ has degree at most d and for all $u, v \in V$, if there is a path from u to v, then

$$\mathrm{dist}(u, v) \leq c \cdot \log |V|.$$

Definition

A d-regular graph G on n vertices is called an (n, d)-graph. Moreover, it is an (n, d, α)-graph if $\lambda(G) \leq \alpha d$.

Theorem

Let $\alpha < 1/2$. Then all (n, d, α) graphs are (c, d)-good for some appropriate $c \in \mathbb{N}$.

Random Walks

A vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ is a probability (distribution) vector if all $p_i \geq 0$ and $\sum_{i \in [n]} p_i = 1$. The probability vector for the *uniform distribution* is

$$\underline{u} := \frac{\mathbf{1}}{n} = \left\langle \frac{1}{n}, \ldots, \frac{1}{n} \right\rangle.$$

Definition

A random walk on a graph $G = (V, E)$ is a *discrete-time stochastic process* $(X_0, X_1, \ldots)$ taking values in V. The vertex X_0 is sampled from some *initial distribution* on V, and X_{i+1} is chosen *uniformly at random from the neighbors of X_i*.

Normalized Adjacency Matrices

Let G be a d-regular graph with adjacency matrix $A = A(G)$. Then its normalized adjacency matrix is

$$\hat{A} := \frac{1}{d} A.$$

- The matrix $\hat{A}^t$ is the transition matrix of the Markov Chain defined by *random walks of length t*, i.e., $(\hat{A}^t)_{i,j}$ is the probability *a random walk starting at i is at j after t steps*.
- The *stationary distribution* of the random walk on G is the uniform distribution, namely, $u\hat{A} = \hat{A}u = u$.
- If $d = \lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of A, then $1 = \hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_n$ with $\hat{\lambda}_i = \lambda_i / d$ for $i \in [n]$ are the eigenvalues of $\hat{A}$ with the same eigenvectors as of A.

Random Walks on Expander Graphs

Rapid Mixing of Walks

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Theorem

Let G be an (n, d, α)*-graph with normalized adjacency matrix* $\hat{A}$*. Then for any distribution vector p and any positive integer t:*

$$\|\hat{A}^t p - u\|_2 \leq \|p - u\|_2 \alpha^t \leq \alpha^t.$$

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$$\begin{aligned}
\left\|\hat{A}p-u\right\|_2 &= \left\|\hat{A}p-\hat{A}u\right\|_2 = \left\|\hat{A}(p-u)\right\|_2 \\
&= \left\|\sum_{2\le i\le n}\hat{\lambda}_i\beta_i v_i\right\|_2 = \sqrt{\left\langle \sum_{2\le i\le n}\hat{\lambda}_i\beta_i v_i, \sum_{2\le i\le n}\hat{\lambda}_i\beta_i v_i\right\rangle} \\
&= \sqrt{\sum_{2\le i\le n}\hat{\lambda}_i^2\beta_i^2\langle v_i,v_i\rangle} \le \alpha\sqrt{\sum_{2\le i\le n}\beta_i^2\langle v_i,v_i\rangle} \\
&= \alpha\left\|\sum_{2\le i\le n}\beta_i v_i\right\|_2 = \alpha\|p-u\|_2\le\alpha.
\end{aligned}$$

□

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Corollary

For every $\varepsilon > 0$, (n, d, α)*-graphs are* (c, d)*-good with*

$$c = -\frac{3}{2 \log \alpha} + \varepsilon.$$

Probabilistic Algorithms

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Let D be a finite subset of $\mathbb{N}$.

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Problem:	Decide whether $f \neq g$, i.e., $f(a) \neq g(a)$ for some $a \in D$.

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We can do much better by using *randomness*.

RandomPolyTest(f, g)

1. Choose $a \in D$ *uniformly at random*.
2. **if** $f(a) \neq g(a)$ **then** accept.
3. **else** reject.

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Definition

A problem Q is in RP (Randomized Polynomial Time), if there is a *probabilistic polynomial time* algorithm $\mathbb{A}$ such that

$$x \in Q \implies \Pr[\mathbb{A} \text{ accepts } x] \geq \frac{1}{2};$$
$$x \notin Q \implies \Pr[\mathbb{A} \text{ accepts } x] = 0.$$

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$$\begin{aligned} x \in Q &\implies \Pr[\mathbb{A}^2 \text{ accepts } x] \geq \frac{3}{4}; \\ x \notin Q &\implies \Pr[\mathbb{A}^2 \text{ accepts } x] = 0. \end{aligned}$$

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Since randomness is *expensive*, thus the question:

Can we use less than $k \cdot r_x$ many random bits?

(The *ultimate goal* is to use no random bits at all.)

RP Error Reduction (cont'd)

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Theorem (**Ajtai**, **Komlós**, and **Szemerédi**, 1987)

There is a probabilistic polynomial time that achieves the error probability below 2^k *using random bits*

$$r_x + O(k).$$

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- *All $t + 1$ runs of $\mathbb{A}$ falsely reject x.*

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Theorem (**Ajtai**, **Komlós**, and **Szemerédi**, 1987)
Let $G = (V, E)$ *be an* (n, d, α)*-graph and* $B \subseteq V$ *with* $B = \beta n$*. Then*

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Compared to:

$$\Pr[\text{Independently choose } t \text{ many } X\text{s from } V \text{ and all in } B] = \beta^t.$$

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$$P\hat{A}Pw = P\hat{A}u + P\hat{A}z = Pu + P\hat{A}z \quad \text{and} \quad \left\| P\hat{A}Pw \right\|_2 \le \|Pu\|_2 + \|P\hat{A}z\|_2.$$

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The result follows if we can show

$$\|Pu\|_2 \leq \beta\|w\|_2 \quad \text{and} \quad \|P\hat{A}z\|_2 \leq \alpha\|w\|_2.$$

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By $Pw = w$, the support of w, i.e., $\{i \mid w_i \neq 0\}$, has at most βn coordinates. Then, by **Cauchy-Schwartz**,

$$1 = \sum_{i \in [n]} w_i \leq \sqrt{\beta n} \|w\|_2.$$

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$\left\|P\hat{A}z\right\|_2 \leq \alpha\|w\|_2$:

Recall $\langle u, z\rangle = 0$, thus we can write $z = \sum_{2 \leq i \leq n} \gamma_i v_i$.

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Furthermore, $\|Pu\|_2 = \sqrt{\beta / n}$. ⊣

$\left\| P\hat{A}z \right\|_2 \leq \alpha \|w\|_2$:

Recall $\langle u, z \rangle = 0$, thus we can write $z = \sum_{2 \leq i \leq n} \gamma_i v_i$. Then

$$\begin{aligned} \left\| P\hat{A}z \right\|_2 \leq \|\hat{A}z\|_2 = \left\| \sum_{2 \leq i \leq n} \hat{\lambda}_i \gamma_i v_i \right\|_2 \leq \alpha \left\| \sum_{2 \leq i \leq n} \gamma_i v_i \right\|_2 \\ = \alpha \|z\|_2 \leq \alpha \|w\|_2. \end{aligned}$$

⊣

Expansion and Spectral Gap

Recall:

Theorem (**Dodziuk**, 84; **Alon** and **Milman**, 85; **Alon**, 86)

Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

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Note, we cannot replace λ_2 by λ: Consider the complete bipartite graph $K_{2,2}$.

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As $\langle f, \mathbf{1}\rangle = 0$,

$$\lambda_2 \ge \frac{fAf}{\|f\|_2^2} = \frac{nd|S||\bar{S}| - n^2|E(S,\bar{S})|}{n|S||\bar{S}|} = d - \frac{n|E(S,\bar{S})|}{|S||\bar{S}|} \ge d - 2h(G).$$

$\square$