

Expander Graphs and Their Applications (IV)

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Last Lecture

Rapid Mixing of Walks

Lemma

For every probability vector p , $\|\hat{A}p - u\|_2 \leq \|p - u\|_2 \alpha \leq \alpha$.

Theorem

Let G be an (n, d, α) -graph with normalized adjacency matrix \hat{A} . Then for any distribution vector p and any positive integer t :

$$\|\hat{A}^t p - u\|_2 \leq \|p - u\|_2 \alpha^t \leq \alpha^t.$$

Sampling by Random Walks on Expander Graphs

Let $G = (V, E)$ be an (n, d, α) -graph, and $B \subseteq V$ with $|B| = \beta n$.

We carry out the following experiment: We pick $X_0 \in V$ uniformly at random and start from it a random walk X_0, \dots, X_t on G .

(B, t) : this random walk is confined to B , i.e. that $X_i \in B$ for all $i \in [t]$.

Theorem (**Ajtai, Komlós, and Szemerédi, 1987**)

Let $G = (V, E)$ be an (n, d, α) -graph and $B \subseteq V$ with $|B| = \beta n$. Then

$$\Pr [(B, t)] \leq (\beta + \alpha)^t.$$

Expansion and Spectral Gap

Recall:

Theorem (**Dodziuk**, 84; **Alon** and **Milman**, 85; **Alon**, 86)

Let G be a d -regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

$d - \lambda_2 = \lambda_1 - \lambda_2$ is the *spectral gap* of G .

Another view of λ

Lemma

$$\lambda = \max \left\{ \frac{\|Av\|_2}{\|v\|_2} \mid v \text{ and } \mathbf{1} \text{ are orthogonal, i.e., } \langle v, \mathbf{1} \rangle = 0 \right\}.$$

Proof.

By $\langle v, \mathbf{1} \rangle = 0$,

$$v = \sum_{2 \leq i \leq n} \beta_i v_i.$$

Then

$$\|Av\|_2 = \left\| \sum_{2 \leq i \leq n} \lambda_i \beta_i v_i \right\|_2 \leq \lambda \left\| \sum_{2 \leq i \leq n} \beta_i v_i \right\|_2 = \lambda \|v\|_2.$$

The equality is attained when $v = v_2$ or $v = v_n$. □

Another view of λ_2

Lemma

$$\lambda_2 = \max \left\{ \frac{vAv}{\|v\|_2^2} \mid v \text{ and } \mathbf{1} \text{ are orthogonal, i.e., } \langle v, \mathbf{1} \rangle = 0 \right\}.$$

Proof.

By $\langle v, \mathbf{1} \rangle = 0$,

$$v = \sum_{2 \leq i \leq n} \beta_i v_i.$$

Then

$$\begin{aligned} vAv &= \langle v, Av \rangle = \left\langle \sum_{2 \leq i \leq n} \beta_i v_i, \sum_{2 \leq i \leq n} \lambda_i \beta_i v_i \right\rangle \\ &= \sum_{2 \leq i \leq n} \lambda_i \langle \beta_i v_i, \beta_i v_i \rangle \leq \lambda_2 \sum_{2 \leq i \leq n} \langle \beta_i v_i, \beta_i v_i \rangle = \lambda_2 \|v\|_2^2. \end{aligned}$$

The equality is attained when $v = v_2$.

□

Proof of $(d - \lambda_2)/2 \leq h(G)$

Choose $S \subseteq V$ with $|S| \leq n/2$ and

$$h(G) = \frac{|\partial S|}{|S|}.$$

Then let $f := |\bar{S}|1_S - |S|1_{\bar{S}}$. So

$$\|f\|_2^2 = |\bar{S}|^2|S| + |S|^2|\bar{S}| = n|S||\bar{S}|,$$

$$fAf = E(S, S)|\bar{S}|^2 + |E(\bar{S}, \bar{S})||S|^2 - 2|S||\bar{S}||E(S, \bar{S})|.$$

Moreover,

$$E(S, S) = d|S| - |E(S, \bar{S})|, \quad \text{and} \quad E(\bar{S}, \bar{S}) = d|\bar{S}| - |E(S, \bar{S})|.$$

As $\langle f, \mathbf{1} \rangle = 0$,

$$\lambda_2 \geq \frac{fAf}{\|f\|_2^2} = \frac{nd|S||\bar{S}| - n^2|E(S, \bar{S})|}{n|S||\bar{S}|} = d - \frac{n|E(S, \bar{S})|}{|S||\bar{S}|} \geq d - 2h(G).$$

□

The discrete Laplacian Matrices

Let $G = (V, E)$ be a d -regular graph with $A = A(G)$. Then the (discrete) Laplacian of G is the matrix

$$L := \underline{L}_G := dI - A.$$

We fix an *arbitrary orientation* of G . Then $K := \underline{K}_G = (k_{v,e})_{v \in V, e \in E}$ is a $|V| \times |E|$ -matrix with

$$k_{v,e} := \begin{cases} 1, & v \text{ is the tail of } e, \\ -1, & v \text{ is the head of } e, \\ 0, & \text{otherwise (including } e \text{ is a selfloop on } v). \end{cases}$$

Lemma

- ▶ $L = KK^T$.
- ▶ For every $f = (f_v)_{v \in V}$

$$fLf^T = fKK^Tf^T = \|fK\|_2^2 = \sum_{e \text{ with tail } u \text{ and head } v} (f_u - f_v)^2.$$

Proof of $h(G) \leq \sqrt{2d(d - \lambda_2)}$

Let g be the eigenvector corresponding to λ_2 , i.e., $g = v_2$.

Define $f = g^+$ and $V^+ = \text{supp}(f)$, i.e., $f_v = \max\{g_v, 0\}$ for every $v \in V$, and $V^+ := \{v \mid f_v > 0\} = \{v \mid g_v > 0\}$. Moreover, we assume $|V^+| \leq |V|/2$.

It suffices to show

$$fLf^T / \|f\|_2^2 \leq d - \lambda_2 \quad \text{and} \quad h(G)^2 / 2d \leq fLf^T / \|f\|_2^2.$$

$fLf^T / \|f\|_2^2 \leq d - \lambda_2$: For every $x \in V^+$

$$\begin{aligned}(Lf)_x &= df_x - \sum_{y \in V} a_{x,y} f_y = dg_x - \sum_{y \in V^+} a_{x,y} g_y \\ &\leq dg_x - \sum_{y \in V} a_{x,y} g_y = (Lg)_x = (d - \lambda_2)g_x.\end{aligned}$$

As $f_x = 0$ for $x \notin V^+$ we have

$$fLf^T = \sum_{x \in V} f_x (Lf)_x \leq (d - \lambda_2) \sum_{x \in V^+} g_x^2 = (d - \lambda_2) \sum_{x \in V} f_x^2 = (d - \lambda_2) \|f\|_2^2.$$

□

Proof of $h(G) \leq \sqrt{2d(d - \lambda_2)}$ (cont'd)

$h(G)^2/2d \leq fLf^T / \|f\|_2^2$: Let

$$B_f := \sum_{\substack{e \text{ is an edge with} \\ \text{endvertices } x, y}} \left| f_x^2 - f_y^2 \right|$$

and we will show in the following two lemmas.

$$h(G)\|f\|_2^2 \leq B_f \leq \sqrt{2d}\|fK\|_2\|f\|_2.$$

Recall: $fLf^T = \|fK\|_2^2$.

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Proof of $h(G) \leq \sqrt{2d(d - \lambda_2)}$ (cont'd)

Lemma

$$B_f \leq \sqrt{2d} \|fK\|_2 \|f\|_2.$$

Proof.

Let $E' := E \setminus \{e \mid e \text{ a selfloop}\}$.

$$B_f = \sum_E |f_x^2 - f_y^2| = \sum_{E'} |f_x^2 - f_y^2| = \sum_{E'} |f_x + f_y| |f_x - f_y|$$

$$\leq \sqrt{\sum_{E'} (f_x + f_y)^2} \sqrt{\sum_{E'} (f_x - f_y)^2}$$

$$\sqrt{\sum_{E'} (f_x - f_y)^2} = \sqrt{\sum_E (f_x - f_y)^2} = \|fK\|_2.$$

$$\sqrt{\sum_{E'} (f_x + f_y)^2} \leq \sqrt{2 \sum_{E'} (f_x^2 + f_y^2)} \leq \sqrt{2d \sum_V f_x^2} = \sqrt{2d} \|f\|_2.$$



Proof of $h(G) \leq \sqrt{2d(d - \lambda_2)}$ (cont'd)

Lemma

$$h(G)\|f\|_2^2 \leq B_f.$$

Proof.

Reorder the vertices in such a way that *if $x < y$, then $f_x \geq f_y$.*

$$\begin{aligned} B_f &= \sum_{\substack{e \text{ is an edge with} \\ \text{endvertices } x < y}} f_x^2 - f_y^2 = \sum_{\dots} \sum_{i=x}^{y-1} (f_i^2 - f_{i+1}^2) \quad (\text{i.e., } x \leq i \text{ and } y \geq i+1) \\ &= \sum_{i=1}^{n-1} (f_i^2 - f_{i+1}^2) |E([i], [\bar{i}])| = \sum_{i \in V^+} (f_i^2 - f_{i+1}^2) |E([i], [\bar{i}])| \\ &\geq h(G) \sum_{i \in V^+} (f_i^2 - f_{i+1}^2) i \quad (\text{by } |V^+| \leq |V|/2) \\ &= h(G) \sum_{i \in V^+} f_i^2 = h(G)\|f\|_2^2. \end{aligned}$$

□

λ , λ_2 and $h(G)$

Recall: $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$

Theorem (**Dodziuk**, 84; **Alon** and **Milman**, 85; **Alon**, 86)

Let G be a d -regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

Corollary

$(d - \lambda)/2 \leq h(G)$. That is, smaller λ implies bigger expansion.

Does bigger expansion imply smaller λ ?

No! Bipartite graphs. Consequently, we cannot apply, say, Expander Mixing Lemma etc, directly on (combinatorial) expander graphs!

G and \tilde{G}

Let $G = (V, E)$ be a d -regular graph. We define a $(d + 1)$ -regular graph \tilde{G} by *adding a (new) selfloop on every vertex in V* .

- ▶ \tilde{G} is not bipartite.
- ▶ $h(\tilde{G}) = h(G)$.
- ▶ $A(\tilde{G}) = A(G) + I$.
- ▶ The eigenvalues of $A(\tilde{G})$ are $d + 1, \lambda_2 + 1, \dots, \lambda_n + 1$ with the same corresponding eigenvectors v_1, v_2, \dots, v_n .

G and \tilde{G} (cont'd)

Theorem

Let $\{G_i\}_{i \in \mathbb{N}}$ be a family of d -regular expander graphs. Then there exists a constant $\delta < d + 1$ such that $\lambda(\tilde{G}_i) \leq \delta$ for every $i \in \mathbb{N}$.

Proof.

As $h(\tilde{G}_i) = h(G_i) \geq \varepsilon$ for a fixed constant $\varepsilon > 0$, by **Dodziuk, Alon** et.al. we have

$$\lambda_2(\tilde{G}_i) \leq \iota.$$

for a fixed constant $\iota < d + 1$.

If $\lambda(\tilde{G}_i) = \lambda_2(\tilde{G}_i)$, then $\lambda(\tilde{G}_i) \leq \iota$.

If $\lambda(\tilde{G}_i) = -\lambda_n(\tilde{G}_i)$, then $\lambda(\tilde{G}_i) = -(\lambda_n(G_i) + 1) \leq d - 1$.

So we can take $\delta := \max\{\iota, d - 1\}$. □

Bigger expansion of G implies smaller λ of \tilde{G} .

Expander Graphs are Good

Recall:

Corollary

For every $\varepsilon > 0$, (n, d, α) -graphs are (c, d) -good with $c = -3/(2 \log \alpha) + \varepsilon$.

Corollary

Let $\{G_i\}_{i \in \mathbb{N}}$ be a family of d -regular expander graphs. Then G_i s are (c, d) -good for some appropriate $c \in \mathbb{N}$.

Proof.

\tilde{G}_i is a $(n, d + 1, \alpha)$ -graph with a fixed constant $\alpha < 1$.

\tilde{G}_i is $(c, d + 1)$ -good for some fixed $c \in \mathbb{N}$.

G_i is (c, d) -good. □

We can prove the above corollary directly using the definition of $h(G)$.