Expander Graphs and Their Applications (V)

Yijia Chen
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Theorem (**Dodziuk**, 84; **Alon** and **Milman**, 85; **Alon**, 86)

Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

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No! Bipartite graphs. Consequently, we cannot apply, say, Expander Mixing Lemma etc, directly on (combinatorial) expander graphs!

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- The eigenvalues of $A(\tilde{G})$ are $d+1, \lambda_2 + 1, \ldots, \lambda_n + 1$ with the same corresponding eigenvectors $v_1, v_2, \ldots, v_n$.

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We can prove the above corollary directly using the definition of $h(G)$.

The Existence of Expander Graphs

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In particular:

Theorem

For every sufficient large $d \in \mathbb{N}$, there exists a $(d^4, d, 1/4)$-graph.

The Inductive Construction of Expander Graphs

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Theorem

If G is an (n, d, α)-graph, then G^k is an (n, d^k, α^k)-graph.

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- $\varphi(\alpha, \beta) \leq \alpha + \beta$.
- $\varphi(\alpha, \beta) \leq 1 - (1 - \beta^2)(1 - \alpha)/2$.

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By induction hypothesis, G_n^2 is a $(d^{4n}, d^4, 1/4)$-graph. Then by the Zig-Zag Theorem,

$$\lambda(G_{n+1}) \leq \lambda(G_n^2) + \lambda(H) = 1/4 + 1/4 = 1/2.$$

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Definition

$G \textcircled{z} H = (V(G) \times [m], E')$ where

$$\begin{aligned} & \text{there is an edge between } (v, i) \text{ and } (u, j) \\ \iff \quad & \text{there exist some } k, \ell \in [m] \text{ such that} \\ & \quad \text{in } H \text{ there are edges between } i \text{ and } k,\ \ell \text{ and } j \\ & \quad \text{and } e_v^k = e_u^\ell \text{ in } G. \end{aligned}$$

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In total, the normalized adjacency matrix of $G \textcircled{z} H$ is

$$Z := \tilde{B} P \tilde{B}.$$

A Weak Version of the Zig-Zag Theorem

Theorem

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G \circled{z} H$ is an (nm, d^2, φ)-graph with

$$\varphi \leq \alpha + \beta + \beta^2.$$

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Recall we have proved
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Proof.

By $\langle v, \mathbf{1}\rangle = 0$, $v = \sum_{2\le i\le n} \beta_i v_i$. Then

$$|vAv| = |\langle v, Av\rangle| = \left|\left\langle \sum_{2\le i\le n} \beta_i v_i, \sum_{2\le i\le n} \lambda_i \beta_i v_i \right\rangle\right|$$

$$= \left| \sum_{2\le i\le n} \lambda_i \langle \beta_i v_i, \beta_i v_i\rangle \right| \le \lambda \sum_{2\le i\le n} \langle \beta_i v_i, \beta_i v_i\rangle = \lambda \|v\|_2^2.$$

The equality is attained when $v = v_2$ or $v = v_n$. □

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$$\frac{|fZf|}{\|f\|_2^2} \leq \alpha + \beta + \beta^2$$

Define $f^{\|}$ by

$$f^{\|}(x, i) := \frac{\sum_{j \in [m]} f(x, j)}{m}$$

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It is easy to verify that $\left\langle f^{\|}, f^{\perp} \right\rangle = 0$, hence

$$\|f\|_2^2 = \|f^{\|}\|_2^2 + \|f^{\perp}\|_2^2.$$

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$$\left\| fZf \right\| = \left| f\tilde{B}P\tilde{B}f \right| \leq \left| f^{\|}\tilde{B}P\tilde{B}f^{\|} \right| + 2\left| f^{\|}\tilde{B}P\tilde{B}f^{\perp} \right| + \left| f^{\perp}\tilde{B}P\tilde{B}f^{\perp} \right|$$

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Then

$$\begin{aligned}\|fZf\| &\leq \left|f^{\parallel}Pf^{\parallel}\right| + 2\left|f^{\parallel}P\tilde{B}f^{\perp}\right| + \left|f^{\perp}\tilde{B}P\tilde{B}f^{\perp}\right| \\ &\leq \left|f^{\parallel}Pf^{\parallel}\right| + 2\beta\|f^{\parallel}\|_2\|f^{\perp}\|_2 + \beta^2\|f^{\perp}\|_2^2.\end{aligned}$$

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The second inequality uses the fact that P is a permutation and the fact that $|f_1 f_2| \leq \|f_1\|_2 \|f_2\|_2$.

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$$g(v) := \sqrt{m} f^{\|}(v, i) = \frac{\sum_{j \in [m]} f(x, j)}{\sqrt{m}}.$$

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$$f^{\|} P f^{\|} = g \hat{A} g$$

where $\hat{A}$ is the normalized adjacency matrix of G.

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Finally we have

$$\|fZf\| \leq \alpha\|f^{\|}\|_2^2 + 2\beta\|f^{\|}\|_2\|f^{\perp}\|_2 + \beta^2\|f^{\perp}\|_2^2.$$

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$$\|fZf\| \leq \alpha\|f^{\parallel}\|_2^2 + 2\beta\|f^{\parallel}\|_2\|f^{\perp}\|_2 + \beta^2\|f^{\perp}\|_2^2.$$

Recall $\|f\|_2^2 = \|f^{\parallel}\|_2^2 + \|f^{\perp}\|_2^2$,

$$\frac{\|fZf\|}{\|f\|_2^2} \leq \alpha + \beta + \beta^2.$$

□