Expander Graphs and Their Applications (VI)

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Last Lecture

Using *probabilistic method* we can prove that

almost every graph is an expander graph.

In particular:

Theorem

For every sufficient large $d \in \mathbb{N}$, there exists a $(d^4, d, 1/4)$-graph.

Inductive Construction Using the Zig-Zag Product

Let G and H be two graphs. We will define their zig-zag product $G \textcircled{z} H$ in such a way that the following theorem is true.

Theorem (Reingold, Vadhan, and Wigderson, 02)

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G \textcircled{z} H$ is an $(nm, d^2, \varphi(\alpha, \beta))$-graph, where the function φ satisfies the following conditions:

- *If $\alpha < 1$ and $\beta < 1$, then $\varphi(\alpha, \beta) < 1$.*
- $\varphi(\alpha, \beta) \leq \alpha + \beta$.
- $\varphi(\alpha, \beta) \leq 1 - (1 - \beta^2)(1 - \alpha)/2$.

Let H be a $(d^4, d, 1/4)$-graph for a constant d. Then we let

$$G_1 := H^2, \quad \text{and} \quad G_{n+1} := G_n^2 \textcircled{z} H \text{ for } n \geq 1.$$

Theorem

Every graph G_n is a $(d^{4n}, d^2, 1/2)$-graph for all $n \in \mathbb{N}$.

The Definition of the Zig-Zag Product

Let G be an (n, n, α)-graph and H an (m, d, β)-graph. For every vertex $v \in V(G)$ we fix some numbering

$$e_v^1, e_v^2, \dots, e_v^m$$

of the edges incident to v in G.
We also assume

$$V(H) = [m] = \{1, 2, \dots, m\}.$$

Definition

$G ⓩ H = (V(G) \times [m], E')$ where

there is an edge between (v, i) and (u, j)

$\iff$ there exist some $k, \ell \in [m]$ such that

in H there are edges between i and k, ℓ and j

and $e_v^k = e_u^\ell$ in G.

The Adjacency Matrix of the Zig-Zag Product

Every edge between (v, i) and (u, j) in $G \textcircled{z} H$ can be viewed a walk length 3 consisting of three steps:

(i) an edge in the v-th copy in H, i.e., the edge between i and k;

(ii) an edge in G, i.e., the edge $e_v^k = e_u^\ell$;

(iii) an edge in the u-th copy of H, i.e., the edge between ℓ and j.

Let $\hat{B}$ be the normalized adjacency matrix of H. (i) and (iii) are done on n disjoint copies of H with the corresponding transition matrix

$$\tilde{B} := \hat{B} \otimes I_n \qquad \textit{(tensor product)}$$

In step (ii) we move from a vertex (v, k) to the *unique* vertex (u, ℓ) with $e_v^k = e_u^\ell$. Thus, the corresponding transition matrix P is defined by

$$P_{(v,k),(u,\ell)} := \begin{cases} 1, & \text{if } e_v^k = e_u^\ell \\ 0, & \text{otherwise.} \end{cases}$$

In total, the normalized adjacency matrix of $G \textcircled{z} H$ is $Z := \tilde{B} P \tilde{B}$.

A Weak Version of the Zig-Zag Theorem

Theorem

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G \textcircled{z} H$ is an (nm, d^2, φ)-graph with

$$\varphi \leq \alpha + \beta + \beta^2.$$

A Refined Analysis of $\varphi(\alpha, \beta)$

Theorem

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G ⓩ H$ is an $(nm, d^2, \varphi(\alpha, \beta))$-graph, where

$$\varphi(\alpha, \beta) = \frac{1}{2}(1 - \beta^2)\alpha + \frac{1}{2}\sqrt{(1 - \beta^2)^2\alpha^2 + 4\beta^2}.$$

Some consequences:

- $\varphi(\lambda, 0) = \varphi(0, \lambda) = \lambda$ and $\varphi(\lambda, 1) = \varphi(1, \lambda) = 1$ for all $0 \leq \lambda \leq 1$.
- $\varphi(\alpha, \beta)$ is a strictly increasing function of both α and β (except when one of them is 1).
- If $\alpha < 1$ and $\beta < 1$, the $\varphi(\alpha, \beta) < 1$.
- $\varphi(\alpha, \beta) \leq \alpha + \beta$ for all $0 \leq \alpha, \beta \leq 1$.

Claim. P is a *reflection* through a linear subspace S of $\mathbb{R}^{mn}$. Hence, for every mn-vector f

$$\frac{\langle Pf, f \rangle}{\|f\|_2^2} = \cos 2\theta,$$

where θ is the angle between f and S. ⊣

Proof.

Recall for all vectors f_1 and f_2 we have $\cos\gamma = \langle f_1, f_2 \rangle / \|f_1\|_2 \|f_2\|_2$ where γ is the angle between f_1 and f_2.

Note $P^2 = I$, the only eigenvalues of P are ± 1. Choose $s \in [mn]$ such that $\lambda_1(P) = \ldots = \lambda_s(P) = 1$ and $\lambda_{s+1}(P) = \ldots = \lambda_{mn}(P) = -1$. Then let S be the subspace of $\mathbb{R}^{mn}$ spanned by $v_1, \ldots, v_s$.

Assume $f = \sum_{i \in [mn]} a_i v_i$. Then

$$Pf = \sum_{i \in [s]} a_i v_i - \sum_{i=s+1}^{mn} a_i v_i.$$

with $\sum_{i \in [s]} a_i v_i \in S$.

□

$$\text{Recall:}\quad \varphi(\alpha,\beta) = \max\left\{ \frac{|fZf|}{\|f\|_2^2} \,\middle|\, \langle f, \mathbf{1}\rangle = 0 \right\} = \max\left\{ \frac{\left|f\tilde{B}P\tilde{B}f\right|}{\|f\|_2^2} \,\middle|\, \langle f, \mathbf{1}\rangle = 0 \right\}$$

$$= \max\left\{ \left|\left\langle f\tilde{B}, P\tilde{B}f \right\rangle\right| / \|f\|_2^2 \,\middle|\, \langle f, \mathbf{1}\rangle = 0 \right\}$$

Furthermore, we decomposed $f = f^{\parallel} + f^{\perp}$ with $\left\langle f^{\parallel}, f^{\perp} \right\rangle = 0$. And $\tilde{B}f^{\parallel} = f^{\parallel}$.

Thus, for $\langle f, \mathbf{1}\rangle = 0$

$$\frac{\left|\left\langle P\tilde{B}f, \tilde{B}f \right\rangle\right|}{\|f\|_2^2} = \frac{\left|\left\langle P(f^{\parallel} + \tilde{B}f^{\perp}), f^{\parallel} + \tilde{B}f^{\perp} \right\rangle\right|}{\|f^{\parallel} + f^{\perp}\|_2^2}$$

$$= |\cos 2\theta| \frac{\|f^{\parallel} + \tilde{B}f^{\perp}\|_2^2}{\|f^{\parallel} + f^{\perp}\|_2^2} = |\cos 2\theta| \frac{\cos^2\phi}{\cos^2\phi'},$$

where θ is the angle between $f^{\parallel} + \tilde{B}f^{\perp}$ and S, $\phi \in [0, \pi/2]$ is the angle between $f^{\parallel}$ and $f^{\parallel} + f^{\perp}$, and $\phi' \in [0, \pi/2]$ is the angle between $f^{\parallel}$ and $f^{\parallel} + \tilde{B}f^{\perp}$.

$\phi \in [0, \pi/2]$ is the angle between $f^{\parallel}$ and $f^{\parallel} + f^{\perp}$:

$$\cos\phi = \frac{\left\langle f^{\parallel}, f^{\parallel} + f^{\perp} \right\rangle}{\|f^{\parallel}\|_2 \|f^{\parallel} + f^{\perp}\|_2} = \frac{\left\langle f^{\parallel}, f^{\parallel} \right\rangle + \left\langle f^{\parallel}, f^{\perp} \right\rangle}{\|f^{\parallel}\|_2 \|f^{\parallel} + f^{\perp}\|_2} = \frac{\|f^{\parallel}\|_2}{\|f^{\parallel} + f^{\perp}\|_2} \geq 0.$$

$\phi' \in [0, \pi/2]$ is the angle between $f^{\parallel}$ and $f^{\parallel} + \tilde{B} f^{\perp}$:

$$\begin{aligned}
\cos\phi' &= \frac{\left\langle f^{\parallel}, f^{\parallel} + \tilde{B} f^{\perp} \right\rangle}{\|f^{\parallel}\|_2 \|f^{\parallel} + \tilde{B} f^{\perp}\|_2} = \frac{\left\langle f^{\parallel}, f^{\parallel} \right\rangle + \left\langle f^{\parallel}, \tilde{B} f^{\perp} \right\rangle}{\|f^{\parallel}\|_2 \|f^{\parallel} + \tilde{B} f^{\perp}\|_2} \\
&= \frac{\left\langle f^{\parallel}, f^{\parallel} \right\rangle + \left\langle \tilde{B} f^{\parallel}, f^{\perp} \right\rangle}{\|f^{\parallel}\|_2 \|f^{\parallel} + \tilde{B} f^{\perp}\|_2} = \frac{\left\langle f^{\parallel}, f^{\parallel} \right\rangle + \left\langle f^{\parallel}, f^{\perp} \right\rangle}{\|f^{\parallel}\|_2 \|f^{\parallel} + \tilde{B} f^{\perp}\|_2} \\
&= \frac{\|f^{\parallel}\|_2}{\|f^{\parallel} + \tilde{B} f^{\perp}\|_2} \geq 0.
\end{aligned}$$

Let ψ be the angle between $f^{\parallel}$ and S, then $\psi - \phi' \leq \theta \leq \psi + \phi'$. (Recall θ is the angle between $f^{\parallel} + \tilde{B} f^{\perp}$ and S.)

We have proved that $|f^{\|} P f^{\|}| \leq \alpha \|f^{\|}\|_2^2$. Hence,

$$\underline{\mu_1} := |\cos 2\psi| = \frac{\left\langle P f^{\|}, f^{\|} \right\rangle}{\|f^{\perp}\|_2^2} \leq \alpha$$

Also, we have proved $\|\tilde{B} f^{\perp}\|_2 \leq \beta \|f^{\perp}\|_2$, i.e.,

$$\underline{\mu_2} := \frac{\tan \phi'}{\tan \phi} = \frac{\|\tilde{B} f^{\perp}\|_2}{\|f^{\perp}\|_2} \leq \beta.$$

Hence, we want to *maximize*

$$|\cos 2\theta| \frac{\cos^2 \phi}{\cos^2 \phi'}$$

subject to

1. $0 \leq \phi, \phi', \psi \leq \pi/2$.
2. $\psi - \phi' \leq \theta \leq \psi + \phi'$.
3. $\mu_1 = |\cos 2\psi| \leq \alpha$.
4. $\mu_2 = \tan \phi' / \tan \phi \leq \beta$.

Proof of $\varphi(\alpha,\beta) = (1-\beta^2)\alpha/2 + \sqrt{(1-\beta^2)^2\alpha^2 + 4\beta^2}/2$

Case 1. $\phi' \le \min\{\psi, \pi/2 - \psi\}$.

$$\begin{aligned} |\cos 2\theta| &= \max\left\{|\cos 2(\psi+\phi')|, |\cos 2(\psi-\phi')|\right\} \\ &= |\cos 2\psi \cos 2\phi'| + |\sin 2\psi \sin 2\phi'|. \end{aligned}$$

By some trigonometric manipulation

$$\begin{aligned} &|\cos 2\theta| \frac{\cos^2\phi}{\cos^2\phi'} \\ =&\frac{1}{2}\left|(1-\mu_2^2)\cos 2\psi + (1+\mu_2^2)\cos 2\psi \cos 2\phi\right| + \frac{1}{2}\left|2\mu_2 \sin 2\psi \sin 2\phi\right| \end{aligned}$$

To maximize the above value, we choose ϕ in such a way that $(\cos 2\phi, \sin 2\phi)$ be a unit vector in the direction of $(\pm(1+\mu_2^2)\cos 2\psi, 2\mu_2 \sin 2\psi)$. Then

$$\begin{aligned} |\cos 2\theta| \frac{\cos^2\phi}{\cos^2\phi'} &\le \frac{1}{2}(1-\mu_2^2)|\cos 2\psi| + \frac{1}{2}\sqrt{(1+\mu_2^2)^2\cos^2 2\psi + 4\mu_2^2 \sin^2 2\psi} \\ &= \frac{1}{2}(1-\mu_2^2)\mu_1 + \frac{1}{2}\sqrt{(1+\mu_2^2)^2\mu_1^2 + 4\mu_2^2(1-\mu_1^2)}. \end{aligned}$$

Proof of $\varphi(\alpha, \beta) = (1-\beta^2)\alpha/2 + \sqrt{(1-\beta^2)^2\alpha^2 + 4\beta^2}/2$

Case 2. $\phi' > \min\{\psi, \pi/2 - \psi\}$.

$$|\cos 2\theta| \frac{\cos^2\phi}{\cos^2\phi'} \le \frac{\cos^2\phi}{\cos^2\phi'} = \mu_2^2 + (1-\mu_2^2)\cos^2\phi.$$

The condition $\phi' > \min\{\psi, \pi/2 - \psi\}$ implies $\cos 2\phi' < |\cos 2\psi| - \mu_1$.
By some trigonometric manipulation

$$\cos 2\phi' = \frac{(1+\mu_2^2)\cos^2\phi - \mu_2^2}{(1-\mu_2^2)\cos^2\phi + \mu_2^2}.$$

$\cos 2\phi' < \mu_1$ is equivalent to

$$\cos^2\phi < \frac{\mu_2^2(1+\mu_1)}{(1-\mu_1) + \mu_2^2(1+\mu_1)}.$$

We conclude

$$|\cos 2\theta| \frac{\cos^2\phi}{\cos^2\phi'} < \frac{2\mu_2^2}{1-\mu_1+\mu_2^2(1+\mu_1)}.$$

□