Expander Graphs and Their Applications (VII)

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Last Lecture

The Zig-Zag Theorem

Theorem

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G \text{ⓩ} H$ is an $(nm, d^2, \varphi(\alpha, \beta))$-graph, where

$$\varphi(\alpha, \beta) = \frac{1}{2}(1-\beta^2)\alpha + \frac{1}{2}\sqrt{(1-\beta^2)^2\alpha^2 + 4\beta^2}.$$

Some consequences:

- $\varphi(\lambda, 0) = \varphi(0, \lambda) = \lambda$ and $\varphi(\lambda, 1) = \varphi(1, \lambda) = 1$ for all $0 \leq \lambda \leq 1$.
- $\varphi(\alpha, \beta)$ is a strictly increasing function of both α and β (except when one of them is 1).
- If $\alpha < 1$ and $\beta < 1$, the $\varphi(\alpha, \beta) < 1$.
- $\varphi(\alpha, \beta) \leq \alpha + \beta$ for all $0 \leq \alpha, \beta \leq 1$.

Corollary

If G is an (n, m, α)*-graph and H an* (m, d, β)*-graph. Then*

$$1 - \hat{\lambda}(G \text{ ⓩ } H) \geq \frac{1}{2}(1 - \beta^2)(1 - \alpha)$$

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Proof.

$$\frac{1}{2}\sqrt{(1-\beta^2)^2\alpha^2 + 4\beta^2} \leq \frac{1}{2}\sqrt{(1-\beta^2)^2 + 4\beta^2} = \frac{1}{2}(1+\beta^2) = 1 - \frac{1}{2}(1-\beta^2).$$

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If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then

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Then

$$\begin{aligned}\varphi(\alpha, \beta) &= \frac{1}{2}(1-\beta^2)\alpha + \frac{1}{2}\sqrt{(1-\beta^2)^2\alpha^2 + 4\beta^2} \\ &\leq 1 - \frac{1}{2}(1-\beta^2)(1-\alpha)\end{aligned}$$

□

Lower Bound of $d - \lambda$ in Arbitrary Graphs

Theorem

Let G be a connected, d-regular, and non-bipartite graph with n vertices. Then

$$d - \lambda \geq \frac{1}{n^2}.$$

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Proof. Recall $\lambda_2 = \max_{\langle x,\mathbf{1}\rangle=0} xAx/\|x\|_2^2 = \max_{\langle x,\mathbf{1}\rangle=0,\|x\|_2=1} xAx$.

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$$\begin{aligned}
\lambda_2 &= \max_{\langle x,\mathbf{1}\rangle=0, \|x\|_2=1} \left(\sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} 2x_u x_v + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2 \right) \\
&= \max_{\langle x,\mathbf{1}\rangle=0, \|x\|_2=1} \left(\sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} (x_u^2 + x_v^2 - (x_u - x_v)^2) + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2 \right) \\
&= \max_{\langle x,\mathbf{1}\rangle=0, \|x\|_2=1} \left(\sum_{u \in V} d x_u^2 - \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \text{ and } v}} (x_u - x_v)^2 \right).
\end{aligned}$$

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i.e.,

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Choose some x with $\langle x, \mathbf{1} \rangle = 0$ and $\|x\|_2 = 1$ such that

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(Here we use the fact that the minimum can be attained.)

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x_{u_0} and x_{u_1} have different signs, i.e., $x_{u_0} x_{u_1} < 0$.

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Recall G is connected. Thus, there exists a path

$$u_0 = v_0, v_1, \ldots, v_k = u_1.$$

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Hence,

$$\sum_{i=0}^{k-1} |x_{v_i} - x_{v_{i+1}}| \geq |x_{v_0} - x_{v_k}| \geq \frac{1}{\sqrt{n}}.$$

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&\geq \sum_{i=0}^{k-1} \left(x_{v_i} - x_{v_{i+1}}\right)^2 \\
&\geq \frac{1}{k} \left(\sum_{i=0}^{k-1} \left| x_{v_i} - x_{v_{i+1}} \right| \right)^2 \\
&\geq \frac{1}{k} \left| x_{v_0} - x_{v_k} \right|^2 \\
&\geq \frac{1}{kn} \geq \frac{1}{n^2}.
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Let G be a connected, d-regular, and non-bipartite graph with n vertices. Then

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Proof. Let x be an eigenvector for λ_n with $\|x\|_2 = 1$.

$$\lambda_n = \lambda_n \|x\|_2^2 = xAx = \sum_{u,v \in [n]} a_{u,v} x_u x_v = \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} 2x_u x_v + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2$$

Thus,

$$d + \lambda_n = \sum_{u \in V} d x_i^2 + \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} 2x_u x_v + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2$$

$$= \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} (x_u + x_v)^2 + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2.$$

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- x_1 is positive and has maximum absolute value among all x_i.

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We assume:

- $V = [n] = \{1, 2, \ldots, n\}$.
- x_1 is positive and has maximum absolute value among all x_i.
- Recall that G is *non-bipartite*, so there exist an edge with end vertices $i, j \in [n]$ such that $x_i, x_j \geq 0$ or $x_i, x_j < 0$.

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$$\begin{aligned}
d + \lambda_n &\geq \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} (x_u + x_v)^2 \geq \sum_{i \in [k-1]} (x_i + x_{i+1})^2 \\
&\geq \frac{1}{k-1} \left(\sum_{i \in [k-1]} |x_i + x_{i+1}| \right)^2 \\
&\geq \frac{1}{k-1} \left((x_1 + x_2) + (-x_2 - x_3) + \ldots + (x_{k-1} + x_k) \right)^2 = \frac{(x_1 + x_k)^2}{k-1} \\
&\geq \frac{x_1^2}{n} \geq \frac{\sum_{i \in [n]} x_i^2}{n^2} = \frac{1}{n^2}.
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&\geq \sum_{j \in [i-1]} (x_j + x_{j+1})^2 + x_i^2 \\
&\geq \frac{1}{i} \left(|x_i| + \sum_{j \in [i-1]} |x_j + x_{j+1}| \right)^2 \\
&\geq \frac{1}{i} \left((x_1 + x_2) + (-x_2 - x_3) + \ldots + (-x_{i-1} - x_i) + x_i \right)^2 \\
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$$\begin{aligned} d + \lambda_n &\geq \frac{1}{i} \left(|x_i| + \sum_{j \in [i-1]} |x_j + x_{j+1}| \right)^2 \\ &\geq \frac{1}{i} \left((x_1 + x_2) + (-x_2 - x_3) + \ldots + (x_{i-1} + x_i) - x_i \right)^2 \\ &\geq \frac{x_1^2}{i} \geq \frac{\sum_{i \in [n]} x_i^2}{n^2} = \frac{1}{n^2}. \end{aligned}$$

□

Reingold's Algorithm

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An important fact: If we design an algorithm by merging several subprograms, each using logspace, then the whole program can use only logspace.

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For a proof, see **Papadimitriou***'s Computational Complexity.*

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and for the edge set $E(H^{G,3})$

- for every vertex $v \in V(H)$ and $i \in [\deg(v)]$, there is en edge between (v, i) and $(v, (i + 1) \mod \deg(v))$;
- for every $u, v \in V(H)$ with $u \neq v$ and $i, j \in \mathbb{N}$, there is an edge between (u, i) and (v, j) if $e_u^i = e_v^j$.

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Theorem

- *$G^{H,3}$ is 3-regular or empty.*

Every graph can be made into degree 3 (cont'd)

Theorem

- *$G^{H,3}$ is 3-regular or empty.*
- *For every $u, v \in V(H)$*

$$\begin{aligned} & \textit{there is a path between } u \textit{ and } v \textit{ in } H \\ \iff \quad & \textit{there is a path between } (u,1) \textit{ and } (v,1) \textit{ in } G^{H,3}. \end{aligned}$$

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Theorem

- $G^{H,D}$ *is D-regular or empty.*
- *For every* $u, v \in V$

$$\begin{aligned} & \textit{there is a path between } u \textit{ and } v \textit{ in } H \\ \iff \quad & \textit{there is a path between } (u,1) \textit{ and } (v,1) \textit{ in } G^{H,D}. \end{aligned}$$

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1. Choose some $\ell = O(\log n)$ such that
$$\left(1 - \frac{1}{d^{16} n^2}\right)^{2^\ell} \leq \frac{1}{2}.$$
2. $G_0 \leftarrow G$.
3. **for** $i = 1$ **to** ℓ **do** $G_i \leftarrow (G_{i-1} \circledz H)^8$.
4. Check if s and t are connected in G_ℓ.

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$$\hat{\lambda}(G_{i-1} \circledz H) \leq 1 - 3\big(1 - \hat{\lambda}(G_{i-1})\big)/8 \leq 1 - \big(1 - \hat{\lambda}(G_{i-1})\big)/3.$$

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Case 2. $\hat{\lambda}(G_{i-1}) > 1/2$. By *expansion*, for all $1/2 \leq x \leq 1$ we have $(1 - (1-x)/3)^4 \leq x$. Then

$$\hat{\lambda}(G_i) \leq (1 - (1 - \hat{\lambda}(G_{i-1}))/3)^8 \leq \hat{\lambda}^2(G_{i-1}).$$

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$$C_0 := C \quad \text{and} \quad C_i := (C_{i-1} \textcircled{z} H)^8$$

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Recall

- Let G be a connected, d-regular, and non-bipartite graph with n vertices. Then $d - \lambda \geq 1/n^2$.

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- We choose ℓ in such a way that

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