Expander Graphs and Their Applications (VIII)

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Last Lecture

The Zig-Zag Theorem

Theorem

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G \textcircled{z} H$ is an $(nm, d^2, \varphi(\alpha, \beta))$-graph, where

$$\varphi(\alpha, \beta) = \frac{1}{2}(1 - \beta^2)\alpha + \frac{1}{2}\sqrt{(1 - \beta^2)^2\alpha^2 + 4\beta^2}.$$

Corollary

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then

$$1 - \hat{\lambda}(G \textcircled{z} H) \geq \frac{1}{2}(1 - \beta^2)(1 - \alpha)$$

Lower Bound of $d - \lambda$ in Arbitrary Graphs

Theorem

Let G be a connected, d-regular, and non-bipartite graph with n vertices. Then

$$d - \lambda \geq \frac{1}{n^2}.$$

Reingold's Algorithm for Graph Reachability Problem

Theorem (**Omer Reingold**, 2005)

There is an algorithm that solves the reachability problem on any graph $G = (V, E)$ using space $O(\log |V|)$.

Let H be a $(d^{16}, d, 1/2)$-graph for some constant $d \in \mathbb{N}$.

Input: A d^{16}-regular graph G with $n := |V(G)|$ and two vertices $s, t \in V(G)$.
Output: Is there a path between s and t in G?

1. Choose some $\ell = O(\log n)$ such that

$$\left(1 - \frac{1}{d^{16} n^2}\right)^{2^\ell} \leq \frac{1}{2}.$$

2. $G_0 \leftarrow G$.
3. **for** $i = 1$ **to** ℓ **do** $G_i \leftarrow (G_{i-1} \textcircled{z} H)^8$.
4. Check if s and t are connected in G_ℓ.

The Analysis of Reingold's Algorithm

For all $i \in [\ell]$ and every *connected component* C of G, we define

$$C_0 := C \quad \text{and} \quad C_i := (C_{i-1} \textcircled{z} H)^8$$

It is easy to see

$$G_i := \bigcup_{\substack{C \text{ a connected} \\ \text{component of } G}} C_i$$

Proposition

$$\hat{\lambda}(C_i) \leq \max\left\{\hat{\lambda}^2(C_{i-1}), \frac{1}{2}\right\}.$$

The Analysis of Reingold's Algorithm (cont'd)

Proposition *For every connected component C of G and all $i \in [\ell]$*

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Corollary

For every connected component C of G

$$\hat{\lambda}(C_\ell) \leq \frac{1}{2}.$$

The Base Graphs

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In the following, we

- provide some *explicit construction*,
- and prove their existence by the *probabilistic method*.

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If G is an (n, d, α)-graph, then G^i is an (n, d^i, λ^i)-graph.

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whose (i,j)th entry is $u_i \cdot v_j$ for every $i \in [n_1]$ and $j \in [n_2]$.

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Let A be an $n_1 \times n_1$ matrix and B an $n_2 \times n_2$ matrix, then their tensor product $A \otimes B$ is an $n_1 \cdot n_2 \times n_1 \cdot n_2$ matrix such that for every $(i_1, j_1), (i_2, j_2) \in [n_1] \times [n_2]$

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Lemma

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If α is an eigenvalue for A with corresponding eigenvector $\vec{u}$, and β is an eigenvalue for B with corresponding eigenvector $\vec{v}$, then $\alpha \cdot \beta$ is an eigenvalue for $A \otimes B$ with corresponding eigenvector $\vec{u} \otimes \vec{v}$.

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Theorem

Let A be an $n_2 \times n_2$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_{n_1}$ and corresponding orthogonal eigenvectors $\vec{u}_1, \ldots, \vec{u}_{n_1}$. And let B be an $n_2 \times n_2$ matrix with eigenvalues $\beta_1, \ldots, \beta_{n_2}$ and corresponding orthogonal eigenvectors $\vec{v}_1, \ldots, \vec{v}_{n_2}$. Then $A \otimes B$ has eigenvalue

$$\alpha_1 \cdot \beta_1, \alpha_2 \cdot \beta_1, \ldots, \alpha_{n_1} \cdot \beta_1, \alpha_1 \cdot \beta_2, \ldots, \alpha_{n_1} \cdot \beta_{n_2}$$

with corresponding orthogonal eigenvectors

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If G_i is an (n_i, d_i, α_i) graph for $i = 1, 2$, then $G_1 \otimes G_2$ is an

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Proof.

$$\begin{aligned}\hat{\lambda}(G_1 \otimes G_2) &= \max\left\{\left|\hat{\lambda}_i(G_1) \cdot \hat{\lambda}_j(G_2)\right| \mid i \in [n_1],\, j \in [n_2], \text{ and } (i \neq 1 \text{ or } j \neq 1)\right\} \\ &= \max\left\{\hat{\lambda}(G_1), \hat{\lambda}(G_2)\right\}.\end{aligned}$$

□

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Then AP_q is a graph with vertex set $\mathbb{F}_q^2$ and edge set

$$\{\text{an edge between } (a,b) \text{ and } (c,d) \mid a,b,c,d \in \mathbb{F}_q \text{ and } ac = b + d\}.$$

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Equivalently, we connect the vertex (a,b) to all points on the line

$$L_{a,b} := \{(x,y) \mid y = ax - b\}.$$

The Affine Plane (cont'd)

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Proof. Let $\underline{M}$ be the $q^2 \times q^2$ normalized adjacency matrix of AP_q.

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Proof. Let M be the $q^2 \times q^2$ normalized adjacency matrix of AP_q. The entry of M^2 in row (a, b) and column (a', b') is exactly the number of common neighbors of (a, b) and (a', b') *divided by* q^2, i.e.,

$$\left| L_{a,b} \cap L_{a',b'} \right| / q^2.$$

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- If $a = a'$ and $b = b'$, then $|L_{a,b} \cap L_{a',b'}| = q$

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Thus $\lambda(M) = 1/\sqrt{q}$. □

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Choosing some sufficiently large q, we can get a $(d^4, d, 1/4)$-graph or a $(d^{16}, d, 1/2)$ graph.

The Probabilistic Method

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$\mathbb{P}(k)$ is a random perfect matching on $[2k]$.

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4. Output (V, E).

$\mathbb{R}_d(k)$ is a d-regular graph on vertices $[2k]$.

An Important Warning: $\mathbb{R}_d(k)$ is not uniformly distributed over all d-regular graphs on vertices $[2k]$.

Main Theorem (Restated)

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Theorem

There exists a constant $c > 0$ such that for all sufficiently large $k \in \mathbb{N}$

$$\Pr\left[h(\mathbb{R}_3(k)) \geq c\right] > 0$$

Proof.

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Consider the event

$$E := \text{there exists a subset } S \subseteq V \text{ with } |S| \leq k = |V|/2 \text{ and } |\partial S| \leq c \cdot |S|.$$

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$$\begin{aligned}
\Pr[E] &\leq \sum_{i \in [k]} \binom{2k}{i} \binom{2k-i}{c \cdot i} \Pr\left[\Gamma([i]) \subseteq [i + c \cdot i]\right] \\
&\leq \sum_{i \in [k]} \binom{2k}{i} \binom{2k}{c \cdot i} \Pr\left[\Gamma([i]) \subseteq [i + c \cdot i]\right].
\end{aligned}$$

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Then

$$\begin{aligned}
\prod_{j=1}^{\lceil i/2 \rceil} \frac{i + c \cdot i - 2j + 1}{2k - 2j + 1} &\leq \prod_{j=0}^{\lceil i/2 \rceil - 1} \frac{i + c \cdot i - 2j}{2k - 2j} \\
&= \frac{2^{\lceil i/2 \rceil} \cdot \prod_{j=0}^{\lceil i/2 \rceil - 1} \left(\lceil (i + c \cdot i)/2 \rceil - j \right)}{2^{\lceil i/2 \rceil} \cdot \prod_{j=0}^{\lceil i/2 \rceil - 1} (k - j)} \\
&= \binom{\lceil (i + c \cdot i)/2 \rceil}{\lceil i/2 \rceil} \Big/ \binom{k}{\lceil i/2 \rceil}
\end{aligned}$$

Binary Entropy

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The binary entropy function $H_2 : (0,1) \to \mathbb{R}$ is defined by

$$H_2(p) = -p \log_2 p - (1-p) \log_2 (1-p).$$

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Lemma

For $p \in (0,1)$ and $n \in \mathbb{N}$

$$\binom{n}{\lceil p \cdot n \rceil} \approx 2^{H(p) \cdot n}.$$

Proof. (cont'd)

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We have seen

$$\begin{aligned}
&\Pr\left[\text{there exists an } S \subseteq V \text{ with } |S| \leq k \text{ and } |\partial S| \leq c \cdot |S|\right] \\
\leq &\sum_{i \in [k]} \binom{2k}{i} \binom{2k}{c \cdot i} \Pr\left[\Gamma([i]) \subseteq [i + c \cdot i]\right] \\
\leq &\sum_{i \in [k]} \binom{2k}{i} \binom{2k}{c \cdot i} \binom{\lceil (i + c \cdot i)/2 \rceil}{\lceil i/2 \rceil}^3 \Big/ \binom{k}{\lceil i/2 \rceil}^3
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\end{aligned}$$

Then

$$\begin{aligned}
&\log_2\left(\binom{2k}{i}\binom{2k}{c\cdot i}\binom{\lceil (i+c\cdot i)/2\rceil}{\lceil i/2\rceil}^3 \Big/ \binom{k}{\lceil i/2\rceil}^3\right) \\
\approx\ & H_2(i/2k)\cdot 2k + H_2(c\cdot i/2k)\cdot 2k \\
&+ 3\cdot H_2(i/(i+c\cdot i))\cdot (i+c\cdot i)/2 - 3H_2(i/2k)\cdot k \\
=\ & -\left(H_2(i/2k) - 2H_2(c\cdot i/2k)\right)\cdot k + 3\cdot H_2(1/(1+c))\cdot (i+c\cdot i)/2.
\end{aligned}$$

Binary Entropy (cont'd)

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Lemma
Let $p \in (0, 1/2)$ and $\varepsilon \in (0, 1/2)$. Then

$$H_2(\varepsilon \cdot p) \leq \delta \cdot H_2(p)$$

for $\delta := -4 \cdot \varepsilon \cdot \log \varepsilon$.

Proof.

$$\begin{aligned} H_2(\varepsilon \cdot p) &= -\varepsilon \cdot p \cdot \log(\varepsilon \cdot p) - (1 - \varepsilon \cdot p) \log(1 - \varepsilon \cdot p) \\ &\leq -2 \cdot \varepsilon \cdot p \cdot \log(\varepsilon \cdot p) \\ &\leq 4 \cdot \varepsilon \cdot p \cdot \log \varepsilon \cdot \log p \\ &\leq -4 \cdot \varepsilon \cdot \log \varepsilon \cdot H_2(p). \end{aligned}$$

□

Proof. (cont'd)

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For $c \in (0, 1/2)$, recall

$$\begin{aligned}
&\log_2 \left(\binom{2k}{i} \binom{2k}{c \cdot i} \binom{\lceil (i + c \cdot i)/2 \rceil}{\lceil i/2 \rceil}^3 \Big/ \binom{k}{\lceil i/2 \rceil}^3 \right) \\
\approx\ & -\left(H_2(i/2k) - 2H_2(c \cdot i/2k)\right) \cdot k + 3 \cdot H_2(1/(1+c)) \cdot (i + c \cdot i)/2 \\
\leq\ & -(1 - c') \cdot H_2(i/2k) \cdot k + c' \cdot i,
\end{aligned}$$

for $c' := \max\left\{ -8 \cdot c \cdot \log c, 3 \cdot H_2(1/(1+c)) \cdot (1+c)/2 \right\}$.

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for some appropriate c (independent of k).

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