Expander Graphs and Their Applications (XII)

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Review of the Previous Lecture

Constraints

Definition

Let $V = \{v_1, \ldots, v_n\}$ be a set of variables, and let Σ be a finite alphabet. A *q-ary constraint* $c = (C, i_1, \ldots, i_q)$ consists of

- a q-tuple of indices $i_1, \ldots, i_q \in [n]$,
- and a subset $C \subseteq \Sigma^q$ of "acceptable" values.

An assignment $\mathcal{A} : V \to \Sigma$ *satisfies* c if

$$(\mathcal{A}(v_{i_1}), \ldots, \mathcal{A}(v_{i_q})) \in C.$$

The formula $X_3 \vee \neg X_7 \vee X_9$ can be viewed as a constraint $(C, 3, 7, 9)$ with

$$\{(1,0,0), (1,0,1), (1,1,0), (1,1,1), (0,0,0), (0,0,1), (0,1,1)\}.$$

Constraint Satisfaction Problems (CSP)

CSP

Input:	A system (i.e., set) $\mathscr{C}$ of constraints over a set of variables V and a finite alphabet Σ.
Problem:	Is there an assignment $\mathcal{A} : V \to \Sigma$ that satisfies every constraint?

3Colorability as CSP

Lemma
3Colorability *is reducible to* CSP.

Proof.
Let $\Sigma := \{0, 1, 2\}$. Given a graph $G = (V, E)$, we identify V with the set of variables.
For each edge $e \in E$ with end vertices v_i and v_j, we introduce a constraint

$$c_e := (C_e, i, j)$$

where

$$C_e := \{(a, b) \in \Sigma^2 \mid a \neq b\}.$$

Then

G is 3-colorable $\iff$ there is an assignment satisfying $\{C_e \mid e \in E\}$.

□

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$$= \min_{\mathcal{A} \text{ an assignment}} \frac{\left|\left\{c \in \mathscr{C} \mid \mathcal{A} \text{ satisfies } c\right\}\right|}{|\mathscr{C}|}.$$

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$$\mathscr{C} \text{ is satisfiable} \iff \mathrm{unsat}(\mathscr{C}) = 0;$$

$$\mathscr{C} \text{ is not satisfiable} \iff \mathrm{unsat}(\mathscr{C}) \geq \frac{1}{|\mathscr{C}|}.$$

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Theorem

$\text{NP} = \text{PCP}(\log n, 1)$ *if and only if for some* $\varepsilon > 0$ *the problem* $\text{Gap-CSP}_{q,\Sigma,\epsilon}$ *is* NP*-hard (for some* $q \geq 2$ *and* $|\Sigma| \geq 2$*).*

Constraint Graphs and Operations on Them

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- *The above definition is the rephrase of a CSP with each constraint being binary.*
- *Sometimes, we also use G to refer to (V, E).*

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We have already seen:

Theorem

Given a constraint graph $G = \langle (V, E), \Sigma, \mathscr{C} \rangle$ *with* $|\Sigma| = 3$, *it is* NP*-hard to decide whether* $\mathrm{unsat}(G) = 0$.

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- $\text{size}(G') \leq C_\Sigma \cdot \text{size}(G)$.
- *If* $\text{unsat}(G) = 0$, *then* $\text{unsat}(G') = 0$.
- *If* $\text{unsat}(G) \neq 0$, *then* $\text{unsat}(G') \geq \min(2 \cdot \text{unsat}(G), \alpha_\Sigma)$.

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Theorem

- $\mathrm{unsat}(\mathbb{G}_\Sigma(G)) \geq \alpha_{\Sigma_0}$.
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- $\mathbb{G}_\Sigma$ *is a polynomial time algorithm.*

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Then we will define the following t-th power of G

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$$\text{for all } u' \in \Gamma(u) \text{ and } v' \in \Gamma(v), \quad \sigma(u') = a_{u'} \text{ and } \sigma(v') = b_{v'}$$

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Back to Expander Graphs

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Using Zig-Zag product, we can construct $\{G_k\}_{k\in\mathbb{N}}$ such that:

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Every graph G_k is a $(d^{4k}, d^2, 1/2)$-graph for all $n \in \mathbb{N}$.

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