

Expander Graphs and Their Applications (XV)

Yijia Chen
Shanghai Jiaotong University

Review of the Previous Lecture

Amplification Lemma

Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d -regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$,

$$\text{unsat}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}(G), \frac{1}{t}\right).$$

Amplification Lemma (stronger version)

Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d -regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$, the following hold.

For every $\vec{\sigma} : V \rightarrow \Sigma^{d^{\lceil t/2 \rceil}}$ let $\sigma : V \rightarrow \Sigma$ be defined according to “popular opinion” by setting, for each $v \in V$, $\sigma(v) := a$ such that

$\Pr[\text{a random } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \text{ reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a],$

where $\vec{\sigma}(w)_v \in \Sigma$ denotes the restriction of $\vec{\sigma}(w)$ to v , is maximized over all $a \in \Sigma$.

Then

$$\text{unsat}_{\vec{\sigma}}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right).$$

Proof of the Amplification Lemma

Let $\vec{\sigma} : V \rightarrow \Sigma^{d^{\lceil t/2 \rceil}}$ be an assignment for G^t . Define the assignment $\sigma : V \rightarrow \Sigma$ as before, i.e., according to “popular opinion.”

For every $v \in V$, let X_v be a random variable such that for every $a \in \Sigma$

$$\Pr[X_v = a] = \Pr[\text{a random } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \\ \text{reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a].$$

Then for every $a \in \Sigma$,

$$\Pr[X_v = \sigma(v)] \geq \Pr[X_v = a].$$

Hence,

$$\Pr[X_v = \sigma(v)] \geq \frac{1}{|\Sigma|}.$$

Proof of the Amplification Lemma (cont'd)

Let

$$F := \begin{cases} \{e \in E \mid \sigma \text{ violates } e\} & \text{if } \text{unsat}_\sigma(G) < 1/t \\ \text{an arbitrary subset of the above } \{\dots\} & \\ \text{with } |F| = \lfloor |E|/t \rfloor & \text{otherwise.} \end{cases}$$

Then

$$\Omega\left(\frac{|F|}{|E|}\right) = \min\left(\text{unsat}_\sigma(G), \frac{1}{t}\right)$$

From now on, we fix $\vec{\sigma}$, σ , and F .

Proof of the Amplification Lemma (cont'd)

Let $\mathbf{E} := E(G^t) = E^t$. Recall there is a one-to-one correspondence between every edge $\mathbf{e} \in \mathbf{E}$ and every walk of length t in G .

With some abuse of notation we write $\mathbf{e} = (v_0, v_1, \dots, v_t)$ where $(v_{i-1}, v_i) \in E$ for all $i \in [t]$.

Definition

A walk $\mathbf{e} = (v_0, v_1, \dots, v_t)$ is hit by its i -th edge if

1. $(v_{i-1}, v_i) \in F$, and
2. Both $\vec{\sigma}(v_0)_{v_{i-1}} = \sigma(v_{i-1})$ and $\vec{\sigma}(v_t)_{v_i} = \sigma(v_i)$.

Remark.

- $(v_{i-1}, v_i) \in F$ means that the edge (v_{i-1}, v_i) rejects σ .
- By 2, v_0 has the major opinion of v_{i-1} in G^t , (which implies that *there is a $\lfloor t/2 \rfloor$ -step walk from v_0 to v_{i-1}*). Similarly, v_t has the majority opinion of v_i .

Hence, (v_0, v_t) rejects $\vec{\sigma}$ by our definition of G^t .

Proof of the Amplification Lemma (cont'd)

Let

$$I := \left\{ \frac{t}{2} - \sqrt{\frac{t}{2}} < i \leq \frac{t}{2} + \sqrt{\frac{t}{2}} \right\} \subseteq \mathbb{N}$$

be the set of “middle” indices. For each walk \mathbf{e} , we define

$$N(\mathbf{e}) := |\{i \in I \mid \mathbf{e} \text{ is hit by its } i\text{-th edge}\}|.$$

$N(\mathbf{e}) > 0$ implies that \mathbf{e} rejects $\vec{\sigma}$.

Thus $\Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \Pr_{\mathbf{e}}[\mathbf{e} \text{ rejects } \vec{\sigma}] = \text{unsat}_{\vec{\sigma}}(G^t)$.

We will prove

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

Combined with $\Omega(|F|/|E|) = \min(\text{unsat}_{\sigma}(G), 1/t)$,

$$\Omega(\sqrt{t}) \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right) \leq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \text{unsat}_{\vec{\sigma}}(G^t).$$

Proof of the Amplification Lemma (cont'd)

To show:

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

we will prove two lemmas.

Lemma

$$\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Lemma

$$\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Proof of the Amplification Lemma (cont'd)

From probability theory:

Lemma

For every *non-negative* random variable X which is not identically zero,

$$\Pr[X > 0] \geq \frac{\mathbb{E}^2[X]}{\mathbb{E}[X^2]}.$$

Proof.

$$\mathbb{E}[X] = \mathbb{E}[X \cdot \mathbf{1}_{X>0}] \leq \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\mathbb{E}[(\mathbf{1}_{X>0})^2]} = \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\Pr[X > 0]}.$$

□

Now by the previous lemmas

$$\Pr[N(\mathbf{e}) > 0] \geq \mathbb{E}^2[N(\mathbf{e})]/\mathbb{E}[(N(\mathbf{e}))^2] = \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Proof of $\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$

Recall we define $I := \left\{ t/2 - \sqrt{t/2} < i \leq t/2 + \sqrt{t/2} \right\}$. For every $i \in I$ and every $\mathbf{e} \in \mathbf{E}$ we define an indicator variable

$$\underline{N_i(\mathbf{e})} := 1 \iff \text{the walk } \mathbf{e} \text{ is hit by its } i\text{-th edge.}$$

Then for every $\mathbf{e} \in \mathbf{E}$

$$N(\mathbf{e}) = \sum_{i \in I} N_i(\mathbf{e}).$$

And by **the linearity of expectation**

$$\mathbb{E}[N] = \sum_{i \in I} \mathbb{E}[N_i].$$

Proof of $\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

To estimate $\mathbb{E}[N_i]$, we choose $\mathbf{e} \in \mathbf{E}$ according to the following distribution:

1. Choose $e = (u, v) \in E$ uniformly at random.
2. Choose a random walk of length $i - 1$ from u , i.e., $\mathbf{u} = v_{i-1}, v_{i-2}, \dots, v_0$.
3. Choose a random walk of length $t - i$ from v , i.e., $\mathbf{v} = v_i, v_{i+1}, \dots, v_t$.
4. Output the walk $\mathbf{e} = (v_0, \dots, v_t)$.

This yields the uniform distribution on \mathbf{E} .

Proof of $\mathbb{E}_e[N(e)] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

For the step

1. Choose $e = (u, v) \in E$ uniformly at random.

$$\Pr[e \in F] = \frac{|F|}{|E|}.$$

For the steps:

- 2 Choose a random walk of length $i - 1$ from u , i.e., $u = v_{i-1}, v_{i-2}, \dots, v_0$.

- 3 Choose a random walk of length $t - i$ from v , i.e., $v = v_i, v_{i+1}, \dots, v_t$.

the choice of v_0 only depends on u , and the choice of v_t only depends on v .

Therefore,

$$\Pr[N_i > 0] = \frac{|F|}{|E|} \cdot p_u \cdot p_v$$

where

$$p_u := \Pr_{v_0} [\vec{\sigma}(v_0)_u = \sigma(u)] \quad \text{and} \quad p_v := \Pr_{v_t} [\vec{\sigma}(v_t)_v = \sigma(v)].$$

Proof of $\mathbb{E}_e[N(e)] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

We look at $p_u := \Pr_{v_0} [\vec{\sigma}(v_0)_u = \sigma(u)]$. For every $\ell \in \mathbb{N}$, define a random variable $X_{u,\ell}$ such that for every $a \in \Sigma$

$$\Pr [X_{u,\ell} = a] = \Pr [\text{a random } \ell\text{-step walk in } G \text{ from } u \\ \text{ends in a vertex } w \text{ for which } \vec{\sigma}(w)_u = a].$$

Therefore

$$p_u = \Pr [X_{u,i-1} = \sigma(u)]$$

and

$$\Pr [X_{u,t/2} = \sigma(u)] \geq \frac{1}{|\Sigma|}.$$

So for $i - 1 = t/2$, we have $p_u \geq 1/|\Sigma|$.

Proof of $\mathbb{E}_e[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

We aim to show for all $\ell \in \mathbb{N}$

$$\left| \ell - \frac{t}{2} \right| \leq \sqrt{\frac{t}{2}} \implies \Pr[X_{u,\ell} = \sigma(u)] \geq \frac{\tau}{2} \cdot \Pr[X_{u,t/2} = \sigma(u)]$$

for a constant $\tau > 0$ to be determined later.

Intuitively, the selfloops of G make the distribution of vertices reached by a random $t/2$ -step walk from u roughly the same as distribution on vertices reached by an ℓ -step walk from u , for every $\ell \in I$.

Proof of $\mathbb{E}_e[N(e)] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

Fix some $\ell \in I$. *Mark one selfloop on each vertex.* Observe that every length- ℓ walk from u in G can be equivalently described by

- (i) specifying in which which steps the *marked* edges were traversed; and
- (ii) specifying the remaining steps conditioned on choosing only *non-marked* edges.

Let $X'_{u,k}$ be a random variable such that for every $a \in \Sigma$

$$\Pr[X'_{u,k} = a] = \Pr[\text{a } k\text{-step walk in } G \text{ conditioned on walking only on } \textit{non-marked edges} \text{ reaches a vertex } w \text{ with } \vec{\sigma}(w)_u = a].$$

Proof of $\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

For a binomial variable $B_{\ell,p}$ with

$$\Pr [B_{\ell,p} = k] = \binom{\ell}{k} \cdot p^k \cdot (1-p)^{\ell-k}$$

with $p := 1 - 1/d$.

$$\Pr [X_{u,\ell} = a] = \sum_{k=0}^{\ell} \Pr [B_{\ell,p} = k] \cdot \Pr [X'_{u,k} = a].$$

The main point is that

if $|\ell_1 - \ell_2|$ is small, the the distributions $B_{\ell_1,p}$ and $B_{\ell_2,p}$ are similar.

Proof of $\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$

Lemma

For every $p \in (0, 1)$ and $c > 0$ there exists some $\ell_0 > 0$ and $0 < \tau < 1$ such that for every $\ell_1, \ell_2 \in \mathbb{N}$ with $\ell_0 < \ell_1 - \sqrt{\ell_1} \leq \ell_2 < \ell_1 + \sqrt{\ell_1}$ we have

$$\text{for all } k \in \mathbb{N} \text{ if } |k - p \cdot \ell_1| \leq c \cdot \sqrt{\ell_1}, \text{ then } \tau \leq \frac{\Pr[B_{\ell_1, p} = k]}{\Pr[B_{\ell_2, p} = k]} \leq \frac{1}{\tau}$$

We choose sufficiently large $c > 0$ such that for the set

$$K := \left\{ k \in \mathbb{N} \mid |k - p \cdot t/2| \leq c \cdot \sqrt{t/2} \right\} \quad \text{where } p := 1 - \frac{1}{d}$$

we have

$$\Pr_{k \sim B_{t/2, p}} [k \notin K] = \sum_{k \notin K} \Pr [B_{t/2, p} = k] < \frac{1}{2 \cdot |\Sigma|}.$$

Apply the above lemma on $\ell_1 = t/2$ and $\ell_2 = \ell$: for some $\tau > 0$ and all $k \in K$

$$\Pr [B_{\ell, p} = k] \geq \tau \cdot \Pr [B_{t/2, p} = k].$$

Proof of $\mathbb{E}_e[N(e)] \geq \Omega(\sqrt{t}) \cdot |F|/|E|$

We now have for all $\ell \in I$

$$\begin{aligned} & \Pr[X_{u,\ell} = \sigma(u)] \\ & \geq \sum_{k \in K} \Pr[B_{\ell,p} = k] \cdot \Pr[X'_{u,k} = \sigma(u)] \\ & \geq \tau \cdot \sum_{k \in K} \Pr[B_{t/2,p} = k] \cdot \Pr[X'_{u,k} = \sigma(u)] \\ & \geq \tau \cdot \left(\sum_k \Pr[B_{t/2,p} = k] \cdot \Pr[X'_{u,k} = \sigma(u)] - \sum_{k \notin K} \Pr[B_{t/2,p} = k] \right) \\ & \geq \tau \cdot \left(\Pr[X_{u,t/2} = \sigma(u)] - \frac{1}{2 \cdot |\Sigma|} \right) \geq \frac{\tau}{2} \cdot \Pr[X_{u,t/2} = \sigma(u)] \end{aligned}$$

As both $i - 1$ and $t - i$ are at most $\sqrt{t/2}$ away from $t/2$.

$$p_u, p_v \geq \frac{\tau}{2 \cdot |\Sigma|}.$$

Recall $\Pr[N_i > 0] = |F|/|E| \cdot p_u \cdot p_v$, so $\mathbb{E}[N_i] \geq \Omega(1) \cdot |F|/|E|$. □

Lemma

For every $p \in (0, 1)$ and $c > 0$ there exists some $0 < \tau < 1$ such that for every $n, m \in \mathbb{N}$ with $4 \leq n - \sqrt{n} \leq m < n + \sqrt{n}$ and all $k \in \mathbb{N}$ with $|k - p \cdot n| \leq c \cdot \sqrt{n}$ we have

$$\tau \cdot \Pr [B_{m,p} = k] \leq \Pr [B_{n,p} = k] \leq \frac{\Pr [B_{m,p} = k]}{\tau}$$

Proof.

First assume $m \leq n$ and recall $n - \sqrt{n} \leq m$. Therefore

$$n = m + r \quad \text{for some } 0 \leq r \leq \sqrt{n}.$$

Recall

$$\binom{m+1}{k} = \frac{m+1}{m+1-k} \cdot \binom{m}{k}.$$

Proof. (cont')

$$\begin{aligned} & \Pr [B_{n,p} = k] \\ &= \binom{m+r}{k} \cdot p^k \cdot (1-p)^{m+r-k} \\ &= \frac{m+1}{m+1-k} \cdot \frac{m+2}{m+2-k} \cdots \frac{m+r}{m+r-k} \cdot \binom{m}{k} \cdot p^k \cdot (1-p)^{m-k} \cdot (1-p)^r \\ &= X \cdot p^k \cdot (1-p)^{m-k} \cdot \binom{m}{k} \\ &= X \cdot \Pr [B_{m,p} = k], \end{aligned}$$

where

$$X = (1-p)^r \cdot \frac{m+1}{m+1-k} \cdot \frac{m+2}{m+2-k} \cdots \frac{m+r}{m+r-k}.$$

In the following, we bound X .

Proof. (cont'd)

For every $a \leq r \leq \sqrt{n}$ we let

$$X_a = \frac{m+a}{m+a-k} = \frac{1}{1-k/(m+a)}.$$

Observe

$$\frac{1}{1-k/m} \geq \frac{1}{1-k/(m+a)} \geq \frac{1}{1-k/n}.$$

Since $k \geq p \cdot n - c \cdot \sqrt{n}$,

$$X_a \geq \frac{1}{1-k/n} \geq \frac{1}{1-p+c/\sqrt{n}} = \frac{1}{1-p} \cdot \frac{1}{1+(c/(1-p)) \cdot (1/\sqrt{n})}.$$

Now let

$$\tau_1 := \left(1 + \frac{c}{1-p} \cdot \frac{1}{\sqrt{n}}\right)^{-r} \leq (1-p)^r \cdot \prod_{a=1}^r X_a = X.$$

Proof. (cont'd)

Similarly, since $n - \sqrt{n} \leq m$, $p \cdot n + c \cdot \sqrt{n} \leq k$, $p \leq 1$, and $n \geq 4$

$$\frac{k}{m} \leq \frac{p \cdot n + c \cdot \sqrt{n}}{n - \sqrt{n}} \leq p + \frac{p \cdot \sqrt{n} + c \cdot \sqrt{n}}{n - \sqrt{n}} \leq p + \frac{2 \cdot c + 2}{\sqrt{n}}.$$

Hence,

$$1 - \frac{k}{m} \geq 1 - p - \frac{2 \cdot c + 2}{\sqrt{n}}.$$

Then

$$X_a \leq \frac{1}{1 - k/m} \leq \frac{1}{1 - p - (2 \cdot c + 2)/\sqrt{n}}$$

Let

$$\tau_2 := \left(\frac{1}{1 - p - (2 \cdot c + 2)/\sqrt{n}} \right)^{-r} \geq (1 - p)^r \cdot \prod_{a=1}^r X_a = X.$$

Proof. (cont'd)

Recall

$$\tau_1 := \left(1 + \frac{c}{1-p} \cdot \frac{1}{\sqrt{n}}\right)^{-r} \leq X$$

$$\tau_2 := \left(\frac{1}{1-p - (2 \cdot c + 2)/\sqrt{n}}\right)^{-r} \geq X.$$

Since $r \leq \sqrt{n}$, both τ_1 and τ_2 can be bounded by constants which only depend on c and p .

We let

$$\tau := \min(\tau_1, 1/\tau_2).$$

Proof. (cont'd)

Finally, if $n < m$, recall $n - \sqrt{n} \leq m < n + \sqrt{n}$, then

$$m - \sqrt{m} < n + \sqrt{n} - \sqrt{n} = n < m.$$

We can deduce the result using the above argument with the roles of m and n reversed, the same p , and

$$c' := c + 1.$$

□