

Expander Graphs and Their Applications (XVI)

Yijia Chen
Shanghai Jiaotong University

Review of the Previous Lecture

Amplification Lemma (stronger version)

Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d -regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$, the following hold.

For every $\vec{\sigma} : V \rightarrow \Sigma^{d^{\lceil t/2 \rceil}}$ let $\sigma : V \rightarrow \Sigma$ be defined according to “popular opinion” by setting, for each $v \in V$, $\sigma(v) := a$ such that

$\Pr[\text{a random } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \text{ reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a],$

where $\vec{\sigma}(w)_v \in \Sigma$ denotes the restriction of $\vec{\sigma}(w)$ to v , is maximized over all $a \in \Sigma$.

Then

$$\text{unsat}_{\vec{\sigma}}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right).$$

Proof of the Amplification Lemma

Let $\vec{\sigma} : V \rightarrow \Sigma^{d^{\lceil t/2 \rceil}}$ be an assignment for G^t . Define the assignment $\sigma : V \rightarrow \Sigma$ as before, i.e., according to “popular opinion.”

For every $v \in V$, let X_v be a random variable such that for every $a \in \Sigma$

$$\Pr[X_v = a] = \Pr[\text{a } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \\ \text{reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a].$$

Then for every $a \in \Sigma$,

$$\Pr[X_v = \sigma(v)] \geq \Pr[X_v = a].$$

Hence,

$$\Pr[X_v = \sigma(v)] \geq \frac{1}{|\Sigma|}.$$

Proof of the Amplification Lemma (cont'd)

Let

$$F := \begin{cases} \{e \in E \mid \sigma \text{ violates } e\} & \text{if } \text{unsat}_\sigma(G) < 1/t \\ \text{an arbitrary subset of the above } \{\dots\} & \\ \text{with } |F| = \lfloor |E|/t \rfloor & \text{otherwise.} \end{cases}$$

Then

$$\Omega\left(\frac{|F|}{|E|}\right) = \min\left(\text{unsat}_\sigma(G), \frac{1}{t}\right)$$

From now on, we fix $\vec{\sigma}$, σ , and F .

Proof of the Amplification Lemma (cont'd)

Let $\mathbf{E} := E(G^t) = E^t$. Recall there is a one-to-one correspondence between every edge $\mathbf{e} \in \mathbf{E}$ and every walk of length t in G .

With some abuse of notation we write $\mathbf{e} = (v_0, v_1, \dots, v_t)$ where $(v_{i-1}, v_i) \in E$ for all $i \in [t]$.

Definition

A walk $\mathbf{e} = (v_0, v_1, \dots, v_t)$ is hit by its i -th edge if

1. $(v_{i-1}, v_i) \in F$, and
2. Both $\vec{\sigma}(v_0)_{v_{i-1}} = \sigma(v_{i-1})$ and $\vec{\sigma}(v_t)_{v_i} = \sigma(v_i)$.

Remark.

- $(v_{i-1}, v_i) \in F$ means that the edge (v_{i-1}, v_i) rejects σ .
- By 2, v_0 has the major opinion of v_{i-1} in G^t , (which implies that *there is a $\lfloor t/2 \rfloor$ -step walk from v_0 to v_{i-1}*). Similarly, v_t has the majority opinion of v_i .

Hence, (v_0, v_t) rejects $\vec{\sigma}$ by our definition of G^t .

Proof of the Amplification Lemma (cont'd)

Let

$$I := \left\{ \frac{t}{2} - \sqrt{\frac{t}{2}} < i \leq \frac{t}{2} + \sqrt{\frac{t}{2}} \right\} \subseteq \mathbb{N}$$

be the set of “middle” indices. For each walk \mathbf{e} , we define

$$N(\mathbf{e}) := |\{i \in I \mid \mathbf{e} \text{ is hit by its } i\text{-th edge}\}|.$$

$N(\mathbf{e}) > 0$ implies that \mathbf{e} rejects $\vec{\sigma}$.

Thus $\Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \Pr_{\mathbf{e}}[\mathbf{e} \text{ rejects } \vec{\sigma}] = \text{unsat}_{\vec{\sigma}}(G^t)$.

We will prove

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

Combined with $\Omega(|F|/|E|) = \min(\text{unsat}_{\sigma}(G), 1/t)$,

$$\Omega(\sqrt{t}) \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right) \leq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \text{unsat}_{\vec{\sigma}}(G^t).$$

Proof of the Amplification Lemma (cont'd)

To show:

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

we will prove two lemmas.

Lemma

$$\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Lemma

$$\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Proof of the Amplification Lemma (cont'd)

From probability theory:

Lemma

For every *non-negative* random variable X which is not identically zero,

$$\Pr[X > 0] \geq \frac{\mathbb{E}^2[X]}{\mathbb{E}[X^2]}.$$

Proof.

$$\mathbb{E}[X] = \mathbb{E}[X \cdot \mathbf{1}_{X>0}] \leq \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\mathbb{E}[(\mathbf{1}_{X>0})^2]} = \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\Pr[X > 0]}.$$

□

Now by the previous lemmas

$$\Pr[N(\mathbf{e}) > 0] \geq \mathbb{E}^2[N(\mathbf{e})]/\mathbb{E}[(N(\mathbf{e}))^2] = \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Proof of $\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot |F|/|E|$

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For a walk \mathbf{e} , let \underline{e}_i denote its i -th edge, i.e., $\mathbf{e}_i = (v_{i-1}, v_i)$.

Proof of $\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot |F|/|E|$

For a walk \mathbf{e} , let \mathbf{e}_i denote its i -th edge, i.e., $\mathbf{e}_i = (v_{i-1}, v_i)$. To upper bound $\mathbb{E}[N^2]$ we define a random variable

$$\underline{Z}(\mathbf{e}) := \left| \{i \in I \mid \mathbf{e}_i \in F\} \right|,$$

i.e., $Z(\mathbf{e})$ counts how many times \mathbf{e} intersects F in the *middle* portion I .

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For every $i \in I$, let

$$\underline{Z}_i(\mathbf{e}) := 1 \quad \iff \quad \mathbf{e}_i \in F.$$

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Therefore $Z(\mathbf{e}) = \sum_{i \in I} Z_i(\mathbf{e})$.

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$$\underline{Z}_i(\mathbf{e}) := 1 \iff \mathbf{e}_i \in F.$$

Therefore $Z(\mathbf{e}) = \sum_{i \in I} Z_i(\mathbf{e})$. And

$$\begin{aligned} \mathbb{E}_{\mathbf{e}}[Z(\mathbf{e})^2] &= \sum_{i,j \in I} \mathbb{E}_{\mathbf{e}}[Z_i(\mathbf{e}) \cdot Z_j(\mathbf{e})] \\ &= \sum_{i \in I} \mathbb{E}[Z_i] + 2 \cdot \sum_{i,j \in I, i < j} \mathbb{E}[Z_i \cdot Z_j] = |I| \cdot \frac{|F|}{|E|} + 2 \cdot \sum_{i,j \in I, i < j} \mathbb{E}[Z_i \cdot Z_j] \end{aligned}$$

Proof of $\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

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We will prove:

Proposition *Let $i, j \in I$ with $i < j$ and $F \subseteq E$. Then*

$$\mathbb{E}[Z_i \cdot Z_j] \leq \frac{|F|}{|E|} \cdot \left(\frac{|F|}{|E|} + \left(\frac{\lambda}{d} \right)^{j-i-1} \right).$$

Proof of $\mathbb{E}_{\mathbf{e}}[(N(\mathbf{e}))^2] \leq O(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

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By this proposition

$$\begin{aligned} \mathbb{E}_{\mathbf{e}} [Z(\mathbf{e})^2] &= |I| \cdot \frac{|F|}{|E|} + 2 \cdot \sum_{i, j \in I, i < j} \mathbb{E}[Z_i \cdot Z_j] \\ &\leq O(\sqrt{t}) \cdot \frac{|F|}{|E|} + 2 \cdot \frac{|F|}{|E|} \cdot \sum_{i, j \in I, i < j} \left(\frac{|F|}{|E|} + \left(\frac{\lambda}{d} \right)^{j-i-1} \right) \\ &\leq O(\sqrt{t}) \cdot \frac{|F|}{|E|} + |I|^2 \cdot \left(\frac{|F|}{|E|} \right)^2 + 2 \cdot |I| \cdot \frac{|F|}{|E|} \cdot \sum_{i=0}^{\sqrt{2t}} (\lambda/d)^i \\ &= O(\sqrt{t}) \cdot \frac{|F|}{|E|}. \quad \square \end{aligned}$$

Proof of $\mathbb{E}[Z_i \cdot Z_j] \leq |F|/|E| \cdot (|F|/|E| + (\lambda/d)^{j-i-1})$

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Observe that $Z_i, Z_j \in \{0, 1\}$ and

$$\Pr[Z_j = 1] = \Pr[Z_i = 1] = \frac{|F|}{|E|}.$$

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Thus

$$\begin{aligned}\mathbb{E}[Z_i \cdot Z_j] &= \Pr[Z_i \cdot Z_j = 1] = \Pr[Z_i = 1] \cdot \Pr[Z_j = 1 | Z_i = 1] \\ &= \frac{|F|}{|E|} \cdot \Pr[Z_j = 1 | Z_i = 1].\end{aligned}$$

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We will need the following result:

Theorem

Let $G = (V, E)$ be a d -regular graph with $\lambda(G) \leq \lambda$. Let $F \subseteq E$ be a set of edges without selfloops, and let K be the distribution on vertices induced by selecting a random edge in F and then a random end vertex.

The probability that a random walk that starts with distribution K takes the $(i + 1)$ -th step in F is upper bounded by

$$\frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i.$$

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Assume first $i = 1$ and $j > i$.

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Assume first $i = 1$ and $j > i$. Then by the previous theorem

$$\Pr_{\mathbf{e}} [Z_j(\mathbf{e}) = 1 | Z_1(\mathbf{e}) = 1] \leq \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-2}$$

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Thus

$$\begin{aligned} \Pr_{|\mathbf{e}|=t} [Z_j(\mathbf{e}) = 1 | Z_i(\mathbf{e}) = 1] &= \Pr_{|\mathbf{e}'|=t-i+1} [Z_{j-i+1}(\mathbf{e}') = 1 | Z_1(\mathbf{e}') = 1] \\ &\leq \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-i-1}, \end{aligned}$$

again by the previous theorem.

□

Theorem

Let $G = (V, E)$ be a d -regular graph with $\lambda(G) \leq \lambda$. Let $F \subseteq E$ be a set of edges without selfloops, and let K be the distribution on vertices induced by selecting a random edge in F and then a random end vertex.

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$$\underline{\hat{\lambda}} := \frac{\lambda(G)}{d}.$$

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Let $\underline{\vec{x}}$ be the vector corresponding to the distribution K , i.e., $\underline{\vec{x}} = (x_v)_{v \in V}$ with

$$x_v = \Pr_K[v] = \text{the fraction of edges touching } v \text{ that are in } F, \text{ divided by } 2.$$

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Let $\underline{y_v}$ be the probability that a random step from v is in F . Therefore

$$\underline{\vec{y}} := (y_v)_{v \in V} = \frac{2 \cdot |F|}{d} \cdot \vec{x}.$$

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Now write

$$\vec{x} = \vec{x}^\perp + \vec{x}^\parallel \quad \text{with } \vec{x}^\parallel := \mathbf{1}/n.$$

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Recall \vec{x}^\perp is orthogonal to \vec{x}^\parallel .

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$$\left\| \hat{A}^i \vec{x}^\perp \right\|_2 \leq \hat{\lambda}^i \left\| \vec{x}^\perp \right\|_2 \leq \hat{\lambda}^i \left\| \vec{x} \right\|_2.$$

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Recall \vec{x}^\perp is orthogonal to \vec{x}^\parallel . Thus,

$$\|\hat{A}^i \vec{x}^\perp\|_2 \leq \hat{\lambda}^i \|\vec{x}^\perp\|_2 \leq \hat{\lambda}^i \|\vec{x}\|_2.$$

Observe that $\|\vec{x}\|_2^2 \leq (\sum_{v \in V} |x_v|) \cdot (\max_{v \in V} |x_v|) \leq \max_{v \in V} |x_v| \leq d/(2 \cdot |F|)$.

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Observe that $\left\| \vec{x} \right\|_2^2 \leq \left(\sum_{v \in V} |x_v| \right) \cdot \left(\max_{v \in V} |x_v| \right) \leq \max_{v \in V} |x_v| \leq d/(2 \cdot |F|)$.

$$\langle \vec{y}, \hat{A}^i \vec{x}^\perp \rangle \leq \left\| \vec{y} \right\|_2 \cdot \left\| \hat{A}^i \vec{x}^\perp \right\|_2 \leq \frac{2 \cdot |F|}{d} \cdot \left\| \vec{x} \right\|_2 \cdot \hat{\lambda}^i \cdot \left\| \vec{x} \right\|_2 \leq \hat{\lambda}^i$$

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Now write

$$\vec{x} = \vec{x}^\perp + \vec{x}^\parallel \quad \text{with } \vec{x}^\parallel := \mathbf{1}/n.$$

Recall \vec{x}^\perp is orthogonal to \vec{x}^\parallel . Thus,

$$\|\hat{A}^i \vec{x}^\perp\|_2 \leq \hat{\lambda}^i \|\vec{x}^\perp\|_2 \leq \hat{\lambda}^i \|\vec{x}\|_2.$$

Observe that $\|\vec{x}\|_2^2 \leq (\sum_{v \in V} |x_v|) \cdot (\max_{v \in V} |x_v|) \leq \max_{v \in V} |x_v| \leq d/(2 \cdot |F|)$.

$$\langle \vec{y}, \hat{A}^i \vec{x}^\perp \rangle \leq \|\vec{y}\|_2 \cdot \|\hat{A}^i \vec{x}^\perp\|_2 \leq \frac{2 \cdot |F|}{d} \cdot \|\vec{x}\|_2 \cdot \hat{\lambda}^i \cdot \|\vec{x}\|_2 \leq \hat{\lambda}^i$$

Finally,

$$\langle \vec{y}, \hat{A}^i \vec{x} \rangle \leq \langle \vec{y}, \hat{A}^i \vec{x}^\parallel \rangle + \langle \vec{y}, \hat{A}^i \vec{x}^\perp \rangle \leq \frac{2 \cdot |F|}{d \cdot n} + \hat{\lambda}^i \leq \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i. \quad \square$$