Expander Graphs and Their Applications (XVI)

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Review of the Previous Lecture

Amplification Lemma (stronger version)

Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d-regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$, the following hold.

For every $\vec{\sigma} : V \to \Sigma^{d^{\lceil t/2 \rceil}}$ let $\sigma : V \to \Sigma$ be defined according to "popular opinion" by setting, for each $v \in V$, $\sigma(v) := a$ such that

Pr[*a random $\lceil t/2 \rceil$-step walk in G from v reaches a vertex w with $\vec{\sigma}(w)_v = a$*],

where $\vec{\sigma}(w)_v \in \Sigma$ denotes the restriction of $\vec{\sigma}(w)$ to v, is maximized over all $a \in \Sigma$.
Then

$$\text{unsat}_{\vec{\sigma}}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right).$$

Proof of the Amplification Lemma

Let $\vec{\sigma} : V \to \Sigma^{d^{\lceil t/2 \rceil}}$ be an assignment for G^t. Define the assignment $\sigma : V \to \Sigma$ as before, i.e., according to "popular opinion."

For every $v \in V$, let X_v be a random variable such that for every $a \in \Sigma$

$$\Pr[X_v = a] = \Pr\big[\text{a random } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \text{ reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a\big].$$

Then for every $a \in \Sigma$,

$$\Pr\left[X_v = \sigma(v)\right] \geq \Pr\left[X_v = a\right].$$

Hence,

$$\Pr\left[X_v = \sigma(v)\right] \geq \frac{1}{|\Sigma|}.$$

Proof of the Amplification Lemma (cont'd)

Let

$$F := \begin{cases} \{e \in E \mid \sigma \text{ violates } e\} & \text{if } \mathrm{unsat}_\sigma(G) < 1/t \\ \text{an arbitrary subset of the above } \{\dots\} & \\ \qquad \text{with } |F| = \lfloor |E|/t \rfloor & \text{otherwise.} \end{cases}$$

Then

$$\Omega\left(\frac{|F|}{|E|}\right) = \min\left(\mathrm{unsat}_\sigma(G), \frac{1}{t}\right)$$

From now on, we fix $\vec{\sigma}$, σ, and F.

Proof of the Amplification Lemma (cont'd)

Let $\mathbf{E} := E(G^t) = E^t$. Recall there is a one-to-one correspondence between every edge $\mathbf{e} \in \mathbf{E}$ and every walk of length t in G.
With some abuse of notation we write $\mathbf{e} = (v_0, v_1, \ldots, v_t)$ where $(v_{i-1}, v_i) \in E$ for all $i \in [t]$.

Definition

A walk $\mathbf{e} = (v_0, v_1, \ldots, v_t)$ is hit by its i-th edge if

1. $(v_{i-1}, v_i) \in F$, and
2. Both $\vec{\sigma}(v_0)_{v_{i-1}} = \sigma(v_{i-1})$ and $\vec{\sigma}(v_t)_{v_i} = \sigma(v_i)$.

Remark.

- *$(v_{i-1}, v_i) \in F$ means that the edge (v_{i-1}, v_i) rejects σ.*
- *By 2, v_0 has the major opinion of v_{i-1} in G^t, (which implies that there is a $\lceil t/2 \rceil$-step walk from v_0 to v_{i-1}). Similarly, v_t has the majority opinion of v_i.*

Hence, (v_0, v_t) rejects $\vec{\sigma}$ by our definition of G^t.

Proof of the Amplification Lemma (cont'd)

Let

$$\underline{I} := \left\{ \frac{t}{2} - \sqrt{\frac{t}{2}} < i \leq \frac{t}{2} + \sqrt{\frac{t}{2}} \right\} \subseteq \mathbb{N}$$

be the set of "middle" indices. For each walk $\mathbf{e}$, we define

$$\underline{N(\mathbf{e})} := \left| \{ i \in I \mid \mathbf{e} \text{ is hit by its } i\text{-th edge} \} \right|.$$

$N(\mathbf{e}) > 0$ implies that $\mathbf{e}$ rejects $\vec{\sigma}$.

Thus $\Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \Pr_{\mathbf{e}}[\mathbf{e} \text{ rejects } \vec{\sigma}] = \text{unsat}_{\vec{\sigma}}(G^t)$.

We will prove

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

Combined with $\Omega(|F|/|E|) = \min(\text{unsat}_\sigma(G), 1/t)$,

$$\Omega(\sqrt{t}) \cdot \min\left(\text{unsat}_\sigma(G), \frac{1}{t}\right) \leq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0] \leq \text{unsat}_{\vec{\sigma}}(G^t).$$

Proof of the Amplification Lemma (cont'd)

To show:

$$\Omega(\sqrt{t}) \cdot \frac{|F|}{|E|} \leq \Pr_{\mathbf{e}}[N(\mathbf{e}) > 0].$$

we will prove two lemmas.

Lemma

$$\mathbb{E}_{\mathbf{e}}[N(\mathbf{e})] \geq \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Lemma

$$\mathbb{E}_{\mathbf{e}}\left[\left(N(\mathbf{e})\right)^2\right] \leq O(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Proof of the Amplification Lemma (cont'd)

From probability theory:

Lemma

For every non-negative random variable X which is not identically zero,

$$\Pr[X > 0] \geq \frac{\mathbb{E}^2[X]}{\mathbb{E}[X^2]}.$$

Proof.

$$\mathbb{E}[X] = \mathbb{E}\left[X \cdot 1_{X>0}\right] \leq \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\mathbb{E}\left[(1_{X>0})^2\right]} = \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\Pr[X > 0]}.$$

□

Now by the previous lemmas

$$\Pr[N(\mathbf{e}) > 0] \geq \mathbb{E}^2[N(\mathbf{e})]/\mathbb{E}\left[\left(N(\mathbf{e})\right)^2\right] = \Omega(\sqrt{t}) \cdot \frac{|F|}{|E|}.$$

Proof of $\mathbb{E}_{\mathbf{e}}\left[\left(N(\mathbf{e})\right)^2\right] \le O(\sqrt{t}) \cdot |F|/|E|$

For a walk $\mathbf{e}$, let $\mathbf{e}_i$ denote its i-th edge, i.e., $\mathbf{e}_i = (v_{i-1}, v_i)$. To upper bound $\mathbb{E}[N^2]$ we define a random variable

$$Z(\mathbf{e}) := \left|\left\{ i \in I \mid \mathbf{e}_i \in F \right\}\right|,$$

i.e., $Z(\mathbf{e})$ counts how many times $\mathbf{e}$ intersects F in the *middle* portion I.

$$0 \le N(\mathbf{e}) \le Z(\mathbf{e}) \quad \text{and} \quad \mathbb{E}[N(\mathbf{e})^2] \le \mathbb{E}[Z(\mathbf{e})^2].$$

For every $i \in I$, let

$$Z_i(e) := 1 \quad \Longleftrightarrow \quad \mathbf{e}_i \in F.$$

Therefore $Z(\mathbf{e}) = \sum_{i \in I} Z_i(\mathbf{e})$. And

$$\mathbb{E}_{\mathbf{e}}[Z(\mathbf{e})^2] = \sum_{i,j \in I} \mathbb{E}_{\mathbf{e}}\left[Z_i(\mathbf{e}) \cdot Z_j(\mathbf{e})\right]$$

$$= \sum_{i \in I} \mathbb{E}[Z_i] + 2 \cdot \sum_{i,j \in I, i<j} \mathbb{E}[Z_i \cdot Z_j] = |I| \cdot \frac{|F|}{|E|} + 2 \cdot \sum_{i,j \in I, i<j} \mathbb{E}[Z_i \cdot Z_j]$$

Proof of $\mathbb{E}_{\mathbf{e}}\left[\left(N(\mathbf{e})\right)^2\right] \leq O(\sqrt{t}) \cdot |F|/|E|$ (cont'd)

We will prove:

Proposition *Let $i, j \in I$ with $i < j$ and $F \subseteq E$. Then*

$$\mathbb{E}[Z_i \cdot Z_j] \leq \frac{|F|}{|E|} \cdot \left(\frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-i-1} \right).$$

By this proposition

$$\begin{aligned}
\mathbb{E}_{\mathbf{e}}\left[Z(\mathbf{e})^2\right] &= |I| \cdot \frac{|F|}{|E|} + 2 \cdot \sum_{i,j \in I, i<j} \mathbb{E}[Z_i \cdot Z_j] \\
&\leq O(\sqrt{t}) \cdot \frac{|F|}{|E|} + 2 \cdot \frac{|F|}{|E|} \cdot \sum_{i,j \in I, i<j} \left(\frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-i-1} \right) \\
&\leq O(\sqrt{t}) \cdot \frac{|F|}{|E|} + |I|^2 \cdot \left(\frac{|F|}{|E|}\right)^2 + 2 \cdot |I| \cdot \frac{|F|}{|E|} \cdot \sum_{i=0}^{\sqrt{2t}} (\lambda/d)^i \\
&= O(\sqrt{t}) \cdot \frac{|F|}{|E|}.
\end{aligned}$$

□

Proof of $\mathbb{E}[Z_i \cdot Z_j] \le |F|/|E| \cdot \left(|F|/|E| + (\lambda/d)^{j-i-1}\right)$

Observe that $Z_i, Z_j \in \{0, 1\}$ and

$$\Pr[Z_j = 1] = \Pr[Z_i = 1] = \frac{|F|}{|E|}.$$

Thus

$$\begin{aligned} \mathbb{E}[Z_i \cdot Z_j] &= \Pr[Z_i \cdot Z_j = 1] = \Pr[Z_i = 1] \cdot \Pr\left[Z_j = 1 \middle| Z_i = 1\right] \\ &= \frac{|F|}{|E|} \cdot \Pr\left[Z_j = 1 \middle| Z_i = 1\right]. \end{aligned}$$

Proof of $\mathbb{E}[Z_i \cdot Z_j] \leq |F|/|E| \cdot \left(|F|/|E| + (\lambda/d)^{j-i-1}\right)$ (cont'd)

We will need the following result:

Theorem

Let $G = (V, E)$ be a d-regular graph with $\lambda(G) \leq \lambda$. Let $F \subseteq E$ be a set of edges without selfloops, and let K be the distribution on vertices induced by selecting a random edge in F and then a random end vertex.

The probability that a random walk that starts with distribution K takes the $(i+1)$-th step in F is upper bounded by

$$\frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i .$$

Proof of $\mathbb{E}[Z_i \cdot Z_j] \le |F|/|E| \cdot (|F|/|E| + (\lambda/d)^{j-i-1})$ (cont'd)

Assume first $i = 1$ and $j > i$. Then by the previous theorem

$$\Pr_{\mathbf{e}}\left[Z_j(\mathbf{e}) = 1 \middle| Z_1(\mathbf{e}) = 1\right] \le \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-2}$$

If $j > i > 1$, we don't care where the random walk **e** visited during its first $i-1$ steps, so they can be ignored.
Thus

$$\begin{aligned}\Pr_{|\mathbf{e}|=t}\left[Z_j(\mathbf{e}) = 1 \middle| Z_i(\mathbf{e}) = 1\right] &= \Pr_{|\mathbf{e}'|=t-i+1}\left[Z_{j-i+1}(\mathbf{e}') = 1 \middle| Z_1(\mathbf{e}') = 1\right] \\ &\le \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^{j-i-1},\end{aligned}$$

again by the previous theorem.

□

Theorem

Let $G = (V, E)$ be a d-regular graph with $\lambda(G) \leq \lambda$. Let $F \subseteq E$ be a set of edges without selfloops, and let K be the distribution on vertices induced by selecting a random edge in F and then a random end vertex.

The probability that a random walk that starts with distribution K takes the $(i+1)$-th step in F is upper bounded by

$$\frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i.$$

Proof.

Let $B \subseteq V$ be the *support of K*, i.e.,

$$B := \{v \in V \mid K(v) > 0\}.$$

Let $n := |V|$. Recall $\hat{A}$ is the normalized $n \times n$ adjacency matrix of G. The first and second largest eigenvalues (in absolute value) of $\hat{A}$ are 1 and

$$\hat{\lambda} := \frac{\lambda(G)}{d}.$$

Let $\vec{x}$ be the vector corresponding to the distribution K, i.e., $\vec{x} = (x_v)_{v \in V}$ with

$$x_v = \Pr_K[v] = \text{ the fraction of edges touching } v \text{ that are in } F\text{, divided by 2.}$$

Let y_v be the probability that a random step from v is in F. Therefore

$$\vec{y} := (y_v)_{v \in V} = \frac{2 \cdot |F|}{d} \cdot \vec{x}.$$

Proof. (cont'd)

The probability p equals the probability of landing in B after i steps, and then taking a step in F, i.e.,

$$p = \sum_{v \in B} y_v \left(\hat{A}^i \vec{x}\right)_v = \sum_{v \in V} y_v \left(\hat{A}^i \vec{x}\right)_v = \left\langle \vec{y}, \hat{A}^i \vec{x} \right\rangle .$$

Now write

$$\vec{x} = \vec{x}^{\perp} + \vec{x}^{\parallel} \quad \text{with } \vec{x}^{\parallel} := \mathbf{1}/n.$$

Recall $\vec{x}^{\perp}$ is orthogonal to $\vec{x}^{\parallel}$. Thus,

$$\left\| \hat{A}^i \vec{x}^{\perp} \right\|_2 \leq \hat{\lambda}^i \left\| \vec{x}^{\perp} \right\|_2 \leq \hat{\lambda}^i \left\| \vec{x} \right\|_2 .$$

Observe that $\|\vec{x}\|_2^2 \leq \left(\sum_{v \in V} |x_v|\right) \cdot (\max_{v \in V} |x_v|) \leq \max_{v \in V} |x_v| \leq d/(2 \cdot |F|)$.

$$\left\langle \vec{y}, \hat{A}^i \vec{x}^{\perp} \right\rangle \leq \|\vec{y}\|_2 \cdot \left\| \hat{A}^i \vec{x}^{\perp} \right\|_2 \leq \frac{2 \cdot |F|}{d} \cdot \|\vec{x}\|_2 \cdot \hat{\lambda}^i \cdot \|\vec{x}\|_2 \leq \hat{\lambda}^i$$

Finally,

$$\left\langle \vec{y}, \hat{A}^i \vec{x} \right\rangle \leq \left\langle \vec{y}, \hat{A}^i \vec{x}^{\parallel} \right\rangle + \left\langle \vec{y}, \hat{A}^i \vec{x}^{\perp} \right\rangle \leq \frac{2 \cdot |F|}{d \cdot n} + \hat{\lambda}^i \leq \frac{|F|}{|E|} + \left(\frac{\lambda}{d}\right)^i . \qquad \square$$