

Expander Graphs and Their Applications (XVII)

Yijia Chen
Shanghai Jiaotong University

Review of the Previous Lecture

Amplification Lemma (stronger version)

Lemma

Let $0 < \lambda < d$ and $|\Sigma|$ be constants. Then there exists a constant $\beta_2 = \beta_2(\lambda, d, |\Sigma|) > 0$ such that for every $t \in \mathbb{N}$ and for every d -regular constraint graph G with a selfloop on each vertex and $\lambda(G) \leq \lambda$, the following hold.

For every $\vec{\sigma} : V \rightarrow \Sigma^{d^{\lceil t/2 \rceil}}$ let $\sigma : V \rightarrow \Sigma$ be defined according to “popular opinion” by setting, for each $v \in V$, $\sigma(v) := a$ such that

$\Pr[\text{a random } \lceil t/2 \rceil\text{-step walk in } G \text{ from } v \text{ reaches a vertex } w \text{ with } \vec{\sigma}(w)_v = a],$

where $\vec{\sigma}(w)_v \in \Sigma$ denotes the restriction of $\vec{\sigma}(w)$ to v , is maximized over all $a \in \Sigma$.

Then

$$\text{unsat}_{\vec{\sigma}}(G^t) \geq \beta_2 \cdot \sqrt{t} \cdot \min\left(\text{unsat}_{\sigma}(G), \frac{1}{t}\right).$$

Assignment Tester

Long Code

Let $s \in \mathbb{N}$ and

$$\underline{L} := \{f \mid f : \{0, 1\}^s \rightarrow \{0, 1\}\}.$$

For every string $a = (a_1, \dots, a_s) \in \{0, 1\}^s$ we define $A_a : L \rightarrow \{0, 1\}$ in such a way that for every $f \in L$

$$A_a(f) = f(a).$$

Each A_a itself can be viewed as a string of $|L|$ bits, and the set

$$\{A_a \mid a \in \{0, 1\}^s\}$$

is an error-correcting code called the Long-Code.

Let $\ell \in \mathbb{N}$. $s_1, s_2 \in \{0, 1\}^\ell$ are δ -far (resp. δ -close) from one another if $\text{dist}(s_1, s_2) \geq \delta \cdot \ell$ (resp. if $\text{dist}(s_1, s_2) \leq \delta \cdot \ell$). Recall

$$\text{dist}(s_1, s_2) := \left| \{i \in [\ell] \mid s_1(i) \neq s_2(i)\} \right|.$$

and

$$\text{rdist}(s_1, s_2) := \frac{\text{dist}(s_1, s_2)}{\ell}.$$

Long Code (cont'd)

Lemma

For $a \neq a' \in \{0, 1\}^s$

$$\text{rdist}(A_a, A_{a'}) = \frac{1}{2}.$$

Folded Strings

Let A be a string of length $|L|$, which we also viewed as a function

$$A : L \rightarrow \{0, 1\}.$$

A is *folded over true* if for every $f \in L$

$$A(-f) = -A(f)$$

where $-$ is the Boolean negation.

For a function $\psi : \{0, 1\}^s \rightarrow \{0, 1\}$ the string A is *folded over ψ* if for every $f \in L$

$$A(f) = A(f \wedge \psi).$$

Folded Strings (cont'd)

First, let

$$\underline{L}_\psi := \{f \in L \mid f = f \wedge \psi\}$$

Then we choose some $L'_\psi \subseteq L_\psi$ such that for every $f \in L_\psi$

- ▶ if $-f \in L_\psi$ too, then L'_ψ contains exactly one of f and $-f$;
- ▶ otherwise, L'_ψ contains f .

Lemma

If $A : L \rightarrow \{0, 1\}$ is folded over true and over ψ , then A is uniquely determined by $A \upharpoonright L'_\psi$.

Long-Code Theorem

There exists a **Long-Code Test** \mathbb{T} which is a randomized algorithm that has input a function $\psi : \{0, 1\}^s \rightarrow \{0, 1\}$, and also oracle access to a string $A : L \rightarrow \{0, 1\}$ *folded over true and over ψ* . \mathbb{T} reads the input ψ and tosses some random coins. Based on these it computes a three-bit predicate

$$w : \{0, 1\}^3 \rightarrow \{0, 1\}$$

and three locations $f_1, f_2, f_3 \in L$ in which it queries the string A . It then outputs $w(A(f_1), A(f_2), A(f_3))$. Denote an execution of \mathbb{T} with access to input ψ and string A by $\mathbb{T}^A(\psi)$. Then the following hold.

- (**Perfect Completeness**): If $a \in \{0, 1\}^s$ with $\psi(a) = 1$, then

$$\Pr \left[\mathbb{T}^{A_a}(\psi) \text{ accepts} \right] = 1.$$

- (**Strong Soundness**): For every $\delta \in [0, 1]$ if $A : L \rightarrow \{0, 1\}$ is folded over true and over ψ and at least δ -far from A_a for all $a \in \{0, 1\}^s$ with $\psi(a) = 1$, then

$$\Pr \left[\mathbb{T}^A(\psi) \text{ rejects} \right] \geq \Omega(\delta).$$

Proof of the Long-Code Theorem

Some Technical Changes

Let $s \in \mathbb{N}$. We will identify $\{0, 1\}^s$ with $[n]$ for $n := 2^s$.

We consider Boolean functions $\psi : [n] \rightarrow \{-1, 1\}$ by changing 1 (i.e., true) to -1 , and 0 (i.e., false) to 1.

Long-Code Theorem (restated)

There exists a *Long-Code Test* \mathbb{T} which is a randomized algorithm that has input a function $\psi : [n] \rightarrow \{-1, 1\}$, and also oracle access to a string $A : L \rightarrow \{-1, 1\}$ *folded over true and over ψ* . \mathbb{T} reads the input ψ and tosses some random coins. Based on these it computes a three-bit predicate $w : \{0, 1\}^3 \rightarrow \{-1, 1\}$ and three locations $f_1, f_2, f_3 \in L$ in which it queries the string A . It then outputs $w(A(f_1), A(f_2), A(f_3))$ such that

- **(Perfect Completeness)**: If $a \in [n]$ with $\psi(a) = -1$, then

$$\Pr \left[\mathbb{T}^{A_a}(\psi) \text{ accepts} \right] = 1.$$

- **(Strong Soundness)**: For every $\delta \in [0, 1]$ if $A : L \rightarrow \{-1, 1\}$ is folded over true and over ψ and at least δ -far from A_a for all $a \in \{0, 1\}^s$ with $\psi(a) = -1$, then

$$\Pr \left[\mathbb{T}^A(\psi) \text{ rejects} \right] \geq \Omega(\delta).$$

Standard Definition

We identify $L = \{f \mid [n] \rightarrow \{-1, 1\}\}$ with the **Boolean hypercube** $\{-1, 1\}^n$, and use f, g for the points in the hypercube.

We use letters A, B , or χ to denote functions whose domain is the hypercube.

Standard Definition (cont'd)

For every $A, B : \{-1, 1\}^n \rightarrow \{-1, 1\}$ let

$$\langle A, B \rangle := \mathbb{E}_{f \in L} [A(f) \cdot B(f)] = \frac{\sum_f A(f) \cdot B(f)}{2^n}.$$

For every $\alpha \subseteq [n]$ let

$$\chi_\alpha : \{-1, 1\}^n \rightarrow \{-1, 1\} \quad \text{with} \quad \chi_\alpha(f) := \prod_{i \in \alpha} f(i).$$

Recall $\alpha = \emptyset$ we assume $\prod_{i \in \alpha} f(i) = 1$ for every $f \in L$.

For every $i \in [n]$

$$\chi_{\{i\}} = A_i.$$

Standard Definition (cont'd)

Lemma

For every $\alpha, \beta \subseteq [n]$

- ▶ if $\alpha = \beta$ then $\langle \chi_\alpha, \chi_\beta \rangle = 1$;
- ▶ if $\alpha \neq \beta$ then $\langle \chi_\alpha, \chi_\beta \rangle = 0$.

Standard Definition (cont'd)

Theorem

The characters $\{\chi_\alpha\}$ form an *orthogonal basis* for the space of functions

$$\{A \mid \{-1, 1\}^n \rightarrow \mathbb{R}\}$$

using the inner product defined above.

Therefore, every function $A : \{-1, 1\}^n \rightarrow \{-1, 1\}$ can be written as

$$A = \sum_{\alpha \subseteq [n]} \hat{A}_\alpha \cdot \chi_\alpha \quad \text{where} \quad \hat{A}_\alpha := \langle A, \chi_\alpha \rangle$$

We also have **Parseval's Identity**

$$\sum_{\alpha \subseteq [n]} \hat{A}_\alpha^2 = \langle A, A \rangle = 1.$$

Recall $A : \{-1, 1\}^n \rightarrow \{-1, 1\}$, hence the range of A is a subset of $\{-1, 1\}$.

The Test

Let $\psi : [n] \rightarrow \{-1, 1\}$ be some predicate and

$$\tau := \frac{1}{100}.$$

Let $A : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be a string folded over true and over ψ , i.e., for every $f \in L = \{-1, 1\}^n$

$$A(-f) = -A(f) \quad \text{and} \quad A(f) = A(f \wedge \psi).$$

Note, here $f \wedge \psi$ is defined by

$$(f \wedge \psi)(a) = \begin{cases} -1, & \text{if } f(a) = \psi(a) = -1 \\ 1, & \text{otherwise.} \end{cases}$$

for every $a \in [n]$.

The Test (cont'd)

A function $A : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is the *legal encoding* of the value $a \in [n]$ if $A(f) = f(a)$ for all $f \in L$.

The following procedure tests whether A is close to the legal encoding of some value $a \in [n]$ that satisfies ψ .

1. Select $f, g \in L$ uniformly at random.
2. Set $h := g \cdot \mu$ where $\mu \in L$ is selected by doing the following independently for every $y \in [n]$. If $f(y) = 1$, then set $\mu(y) := -1$. If $f(y) = -1$, then set

$$\mu(y) := \begin{cases} 1 & \text{with probability } 1 - \tau \\ -1 & \text{with probability } \tau. \end{cases}$$

3. Accept unless $A(g) = A(f) = A(h) = 1$.

Proof of the Long-Code Theorem

Theorem (**Completeness**)

If $\alpha \in [n]$ such that $\psi(\alpha) = -1$, then $\Pr[\mathbb{T}^{\alpha}(\psi) \text{ accepts}] = 1$.

Proof.

We fix some $a \in [n]$ for which $\psi(a) = -1$.

Now let $A : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be the string with $A(f) = f(a)$ for all $f \in L$.

Clearly A is folded over true and over ψ .

If $A(f) = f(a) = -1$ then the test accepts.

If $A(f) = f(a) = 1$, then $A(h) = h(a) = -g(a) = -A(g) \neq A(g)$, and again the test accepts.



Proof of the Long-Code Theorem (cont'd)

Theorem (**Soundness**)

There exists a constant $c > 0$ such that for every $\delta \in [0, 1]$, if $A : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is folded over true and over ψ and at least δ -far from A_a for all $a \in [n]$ with $\psi(a) = -1$, then $\Pr [\mathbb{T}^A(\psi) \text{ rejects}] \geq c \cdot \delta$.

Proof of the Soundness

Let us fix some $\delta \in (0, 1]$ and assume that A is δ -far from every A_α for every $a \in [n]$ with $\psi(a) = -1$.

We let

$$\underline{\varepsilon} := \Pr \left[\mathbb{T}^A(\psi) \text{ rejects} \right].$$

We aim to show that

for some appropriate constant c we have $\varepsilon \geq c \cdot \delta$.

Proof of the Soundness (cont'd)

We will prove later:

Proposition. *There exists a constant $C > 0$ such that if \mathbb{T} rejects with probability ε then*

$$\sum_{|\alpha|>1} |\hat{A}_\alpha|^2 \leq C \cdot \varepsilon$$

We will also need:

Theorem (Friedgut, Kalai, and Naor, 2002)

There is a constant $C' > 0$ such that the following holds. Let $n \in \mathbb{N}$, $\rho > 0$, and $A : \{-1, 1\}^n \rightarrow \{-1, 1\}$ with

$$\sum_{|\alpha|>1} |\hat{A}_\alpha|^2 \leq \rho.$$

Then either

$$|\hat{A}_\emptyset|^2 \geq 1 - C' \cdot \rho \quad \text{or} \quad |\hat{A}_{\{i\}}|^2 \geq 1 - C' \cdot \rho \quad \text{for some } i \in [n].$$

Proof of the Soundness (cont'd)

Lemma

1. If A is folded over true, then $\hat{A}_\alpha = 0$ for every $\alpha \subseteq [n]$ with even $|\alpha|$.
2. If A is folded over ψ and for some $i \in \alpha \subseteq [n]$ we have $\psi(i) = 1$, then $\hat{A}_\alpha = 0$.

Proof.

Recall

$$\hat{A}_\alpha = \langle A, \chi_\alpha \rangle = \frac{\sum_f A(f) \cdot \chi_\alpha(f)}{2^n} = \frac{\sum_f (A(f) \cdot \prod_{a \in \alpha} f(a))}{2^n}$$

□

Proof of the Soundness (cont'd)

Then by the previous proposition (yet to be proved) and the theorem of **Friedgut, Kalai, and Naor**, we have some $i \in [n]$ such that

$$\psi(i) = -1 \quad \text{and} \quad \left| \hat{A}_{\{i\}} \right|^2 \geq 1 - C' \cdot C \cdot \varepsilon.$$

Therefore one of the following holds.

- (i) $\hat{A}_{\{i\}} \geq \sqrt{1 - C' \cdot C \cdot \varepsilon} \geq 1 - C' \cdot C \cdot \varepsilon$, or
- (ii) $-\hat{A}_{\{i\}} \geq \sqrt{1 - C' \cdot C \cdot \varepsilon} \geq 1 - C' \cdot C \cdot \varepsilon$

By definition

$$\hat{A}_\alpha = \mathbb{E}_f [A(f) \cdot \chi_\alpha(f)] = 1 - 2 \cdot \text{rdist}(A, \chi_\alpha),$$

so if (i) holds, then

$$\delta \leq \text{rdist}(A, \chi_{\{i\}}) \leq \frac{C \cdot C' \cdot \varepsilon}{2}.$$

We are done.

Proof of the Soundness (cont'd)

Now assume (ii), i.e., $-\hat{A}_{\{i\}} \geq \sqrt{1 - C' \cdot C \cdot \varepsilon} \geq 1 - C' \cdot C \cdot \varepsilon$ holds.
Imagine first that

$$-\hat{A}_{\{i\}} = 1, \quad \text{i.e.,} \quad A = -\chi_{\{i\}} \quad (\text{why?}).$$

Then the probability (over the choice of f , g , and h) that

$$f(i) = g(i) = h(i) = -1$$

is at least $(1 - \tau)/4$. In that case, $A(f) = A(g) = A(h) = 1$ by our tentative assumption $A = -\chi_{\{i\}}$. Then the test \mathbb{T} rejects (so $\varepsilon \geq (1 - \tau)/4 > 1/8$ by $\tau = 1/100$).

This probability can go down by at most $3 \cdot \text{rdist}(A, -\chi_{\{i\}})$, which is an upper bound on the probability that at least one of f , g , h is a point of disagreement between A and $-\chi_{\{i\}}$. Then

$$\varepsilon \geq \frac{1 - \tau}{4} - 3 \cdot \text{rdist}(A, -\chi_{\{i\}}) > \frac{1}{8} - \frac{3}{2} \cdot C \cdot C' \cdot \varepsilon,$$

i.e., $\varepsilon > 1/8 \cdot (1 + 3 \cdot C \cdot C'/2)$.

Proposition. *There exists a constant $C > 0$ such that if \mathbb{T} rejects with probability ε then*

$$\sum_{|\alpha|>1} |\hat{A}_\alpha|^2 \leq C \cdot \varepsilon$$

Proof.

$$\begin{aligned} 1 - \varepsilon &= \Pr[\mathbb{T}^A(\psi) \text{ accepts}] \\ &= \mathbb{E}_{f,g,h} \left[1 - \frac{(1 + A(f))(1 + A(g))(1 + A(h))}{8} \right]. \end{aligned}$$

Since (f, g) and (f, h) are pairs of random independent functions, and $\mathbb{E}[A] = \hat{A}_\emptyset = 0$ by A being folded over true, we have

$$1 - \varepsilon = \frac{7}{8} - \frac{\mathbb{E}_{g,h}[A(g)A(h)]}{8} - \frac{\mathbb{E}_{f,g,h}[A(f)A(g)A(h)]}{8}.$$

Proof. (cont'd)

Using the Fourier Expansion $A(g) = \sum_{\alpha} \hat{A}_{\alpha} \chi_{\alpha}(g)$

$$\mathbb{E}_{g,h} [A(g)A(h)] = \mathbb{E}_{g,h} \left[\sum_{\alpha, \beta \subseteq [n]} \hat{A}_{\alpha} \hat{A}_{\beta} \chi_{\alpha}(g) \chi_{\beta}(h) \right] = \sum_{\alpha \subseteq [n]} \hat{A}_{\alpha}^2 \cdot (-\tau)^{|\alpha|}$$

which is bounded *by τ in absolute value*, since $\hat{A}_{\emptyset} = 0$.

Let $W := \mathbb{E}_{f,g,h} [A(f)A(g)A(h)]$.

$$\begin{aligned} -1 + \tau + 8 \cdot \varepsilon \geq W &= \mathbb{E}_{g,f,\mu} \left[\sum_{\alpha, \beta, \gamma \subseteq [n]} \hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma} \chi_{\alpha}(g) \chi_{\beta}(g\mu) \chi_{\gamma}(f) \right] \\ &= \sum_{\alpha, \gamma \subseteq [n]} \hat{A}_{\gamma} \hat{A}_{\alpha}^2 \mathbb{E}_{f,\mu} [\chi_{\alpha}(\mu) \chi_{\gamma}(f)] \\ &= \sum_{\gamma \subseteq \alpha \subseteq [n]} \hat{A}_{\gamma} \hat{A}_{\alpha}^2 \cdot (-1 + \tau)^{|\gamma|} (-\tau)^{|\alpha \setminus \gamma|}, \end{aligned}$$

the last equality holds by the correlation of μ and f : (i) if $\gamma \not\subseteq \alpha$ then $\mathbb{E}_{f,\mu} [\chi_{\alpha}(\mu) \chi_{\gamma}(f)] = 0$; (ii) $\mathbb{E}[\mu(i)] = \tau$ and $\mathbb{E}[f(i)\mu(i)] = -1 + \tau$ for all $i \in [n]$.

Proof. (cont'd)

We now bound the absolute value of the above term. We claim that

$$\sum_{\gamma \subseteq \alpha} \left((1 - \tau)^{|\gamma|} \tau^{|\alpha \setminus \gamma|} \right)^2 \leq (1 - \tau)^{|\alpha|}.$$

The left hand side is the probability that *tossing $2|\alpha|$ independent τ -biased coins results in a pattern $\gamma\gamma$ where $\gamma \in \{0, 1\}^{|\alpha|}$* . This probability is $(\tau^2 + (1 - \tau)^2)^{|\alpha|} \leq (1 - \tau)^{|\alpha|}$ since $\tau < 1 - \tau$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{\gamma \subseteq \alpha} |\hat{A}_\gamma| (1 - \tau)^{|\gamma|} \tau^{|\alpha \setminus \gamma|} &\leq \sqrt{\sum_{\gamma \subseteq \alpha} |\hat{A}_\gamma|^2} \cdot \sqrt{\sum_{\gamma \subseteq \alpha} ((1 - \tau)^{|\gamma|} \tau^{|\alpha \setminus \gamma|})^2} \\ &\leq (1 - \tau)^{|\alpha|/2}. \end{aligned}$$

Splitting the sum into $|\alpha| = 1$ and $|\alpha| > 1$,

$$|W| \leq \sum_{|\alpha|=1} |\hat{A}_\alpha|^2 (1 - \tau) + \sum_{|\alpha|>1} |\hat{A}_\alpha|^2 (1 - \tau)^{|\alpha|/2}.$$

Proof. (cont'd)

Let $\rho := \sum_{|\alpha| > 1} |\hat{A}_\alpha|^2$. We have $|W| \leq (1 - \rho)(1 - \tau) + \rho(1 - \tau)^{3/2}$, since $\hat{A}_\alpha = 0$ for even $|\alpha|$. Thus

$$\begin{aligned} 1 - \tau - 8\varepsilon &\leq |W| \leq (1 - \tau)((1 - \rho) + \rho\sqrt{1 - \tau}) \\ \implies \rho &\leq \frac{8\varepsilon}{(1 - \tau)(1 - \sqrt{1 - \tau})}. \end{aligned}$$

We are done since τ is a fixed constant.

