

Expander Graphs and Their Applications (XVIII)

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Review of the Previous Lecture

Long-Code Theorem

There exists a **Long-Code Test** \mathbb{T} which is randomized algorithm that has input a function $\psi : \{0, 1\}^s \rightarrow \{0, 1\}$, and also oracle access to a string $A : L \rightarrow \{0, 1\}$ *folded over true and over ψ* . \mathbb{T} reads the input ψ and tosses some random coins. Based on these it computes a three-bit predicate

$$w : \{0, 1\}^3 \rightarrow \{0, 1\}$$

and three locations $f_1, f_2, f_3 \in L$ in which it queries the string A . It then outputs $w(A(f_1), A(f_2), A(f_3))$. Denote an execution of \mathbb{T} with access to input ψ and string A by $\mathbb{T}^A(\psi)$. Then the following hold.

- (**Perfect Completeness**): If $a \in \{0, 1\}^s$ with $\psi(a) = 1$, then

$$\Pr \left[\mathbb{T}^{A_a}(\psi) \text{ accepts} \right] = 1.$$

- (**Strong Soundness**): For every $\delta \in [0, 1]$ if $A : L \rightarrow \{0, 1\}$ is folded over true and over ψ and at least δ -far from A_a for all $a \in \{0, 1\}^s$ with $\psi(a) = 1$, then

$$\Pr \left[\mathbb{T}^A(\psi) \text{ rejects} \right] \geq \Omega(\delta).$$

Assignment Tester

Assignment Tester

Definition

An Assignment Tester with alphabet Σ_0 and reject probability $\varepsilon > 0$ is an algorithm \mathbb{P} whose input is a circuit Φ over Boolean variables in the set X , and whose output is a constraint graph $G = \langle (V, E), \Sigma_0, \mathcal{C} \rangle$ such that $X \subseteq V$ and such that the following hold.

Let $V' := V \setminus X$ and let $a : X \rightarrow \{0, 1\}$ be an assignment.

- **(Completeness)** if a satisfies Φ , then there exists some $b : V' \rightarrow \Sigma_0$ such that $\text{unsat}_{a \cup b}(G) = 0$.
- **(Soundness)** if a does not satisfy Φ , then for all $b : V' \rightarrow \Sigma_0$

$$\text{unsat}_{a \cup b}(G) \geq \varepsilon \cdot \min \{ \text{rdist}(a, s) \mid s : X \rightarrow \{0, 1\} \text{ that satisfies } \Phi \}.$$

Assignment Tester Theorem

Theorem

There is some $\varepsilon > 0$ and an explicit construction of an assignment tester \mathbb{P} with *alphabet* $\Sigma_0 = \{0, 1\}^3$ and rejection probability ε .

The construction

Recall that there is a Long-Code Test \mathbb{T} . We now proceed to construct a system of constraints based on the test \mathbb{T} in *two steps*.

1. (Modified Test:)

Let $X = \{x_1, \dots, x_s\}$ be a set of s Boolean variables. Also, let there be a Boolean variable for each $f \in L'_\psi$.

This can be expanded into an assignment for L which is *folded over true and over ψ* .

Define a modified test \mathbb{T}' : Given an input ψ , oracle access to an assignment $A : L \rightarrow \{0, 1\}$ folded over true and over ψ , and an assignment $\sigma : X \rightarrow \{0, 1\}$, run \mathbb{T} on ψ and A *with probability 1/2*, and otherwise choose a random $x_i \in X$ and a random $f \in L$, and test that $\sigma(x_i) = A(f) \oplus A(f \oplus e_i)$.

The construction (cont'd)

2. (Creating the Constraints:)

Introduce a new variable z_r per outcome r of the coin tosses of \mathbb{T}' .

These variables will take values in $\{0, 1\}^3$, supposedly specifying the correct values of all three variables queried by \mathbb{T}' on coin tosses r .

We construct the following system of constraints:

For every $z_r \in Z$ and a variable y of the three bits accessed by \mathbb{T}' on coin toss r , so $y \in X \cup L$.

This constraint will check that the assignment for z_r would have satisfied \mathbb{T}' , and that it is consistent with the assignment for y .

The construction (cont'd)

The algorithm \mathbb{P} outputs the constraint graph G :

- vertices of G are $X \cup L \cup Z$;
- constraints (and edges) are as specified in (2);
- the alphabet is $\Sigma_0 = \{0, 1\}^3$, where the Boolean variables $X \cup L$ take values only in $\{000, 111\}$ identified with $\{0, 1\}$ (i.e., a constraint involving $y \in X \cup L$ immediately rejects if the value of y is not in $\{000, 111\}$).

Lemma

\mathbb{P} taking $\psi : \{0, 1\}^s \rightarrow \{0, 1\}$ to G is an assignment tester, with $\Sigma_0 = \{0, 1\}^3$ and constant rejection probability $\varepsilon > 0$.

Proof.

We need to show:

- **(Completeness)** If a satisfies ψ , then there exists some $b \in L \cup Z \rightarrow \Sigma_0$ such that $\text{unsat}_{a \cup b}(G) = 0$.
- **(Soundness)** If a does not satisfy ψ , then for all $b \in L \cup Z \rightarrow \Sigma_0$ we have $\text{unsat}_{a \cup b}(G) \geq \varepsilon \cdot \text{rdist}(a, c)$ for every $c : X \rightarrow \{0, 1\}$ which satisfies ψ .

For the completeness, let the assignment for the variables in L be A_g . It is then easy to assign the variables in Z in a consistent manner.

Proof of the Soundness

Let $\sigma : X \rightarrow \{0, 1\}$ and $\delta > 0$ with

$$\text{rdist}(\sigma, c) \geq \delta$$

for every $c : X \rightarrow \{0, 1\}$ which satisfies ψ .

Fix some $b : L \cup Z \rightarrow \Sigma_0$ and let $A := b \upharpoonright L$.

Proposition. $\Pr[\mathbb{T}'^{A, \sigma}(\psi) \text{ rejects}] = \Omega(\delta)$.

Proof. As in (1), we assume A is folded over true and over ψ . Assume first that $A : L \rightarrow \{0, 1\}$ is $\delta/2$ -far from A_a for all a satisfying ψ , then by the *Long-Code Theorem*, \mathbb{T} rejects with probability $\Omega(\delta)$, so does \mathbb{T}' .

Otherwise, A is $\delta/2$ -close to some $A_{a'}$ with a' satisfying ψ . We now compare a' and σ which are both assignments for the variables of ψ . Since a' satisfies ψ

$$\Pr_i[\sigma(x_i) \neq a'(x_i)] = \text{rdist}(\sigma, a') \geq \delta.$$

Proof of the Soundness (cont'd)

Proof of the proposition (cont'd).

Recall: With probability $1/2$, \mathbb{T}' chooses a random i and a random f and checks that $A(f) \oplus A(f \oplus e_i) = \sigma(x_i)$.

A is $\delta/2$ -close to $A_{a'}$, for all $i \in [s]$:

$$\begin{aligned} & \Pr_{f \in L} [A(f) \oplus A(f \oplus e_i) = a'(x_i)] \\ & \geq \Pr_{f \in L} [A(f) = f(a') \text{ and } A(f \oplus e_i) = (f \oplus e_i)(a')] \\ & \geq 1 - 2 \cdot \delta/2 = 1 - \delta. \end{aligned}$$

The check fails if $a'(x_i) \neq \sigma(x_i)$ and yet $A(f) \oplus A(f \oplus e_i) = a'(x_i)$.

Altogether this occurs with probability $(1 - \delta) \cdot \delta \geq \delta/2$. The result follows from the fact that \mathbb{T}' runs this test with probability $1/2$. ⊖

Proof of the Soundness (cont'd)

Consider the assignment $b \upharpoonright Z$. For every random string that causes \mathbb{T}' to reject (on input A, σ), the associated variable z_r is either assigned consistently with A, σ which means that its value immediately causes the associated constraint to reject; or it is inconsistent with A, σ . Each inconsistency will be detected with probability at least $1/3$.

Thus at least $\Omega(\delta)/3$ fraction of the constraints reject. Hence

$$\text{unsat}_{a \cup b}(G) = \frac{\Omega(\delta)}{3}.$$

□.