Expander Graphs and Their Applications (II)

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Last Lecture

Graph Reachability Problem on Good Graphs

For our purpose, let $c, d \in \mathbb{N}$, a graph $G = (V, E)$ is (c, d)-good if each $v \in V$ has degree at most d and for all $u, v \in V$, if there is a path from u to v, then

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Theorem

Let $c, d \in \mathbb{N}$. For (c, d)-good graphs $G = (V, E)$, the reachability problem can be solved in space

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i.e., in LogSpace.

Graph Expansion

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Definition

Let $d \in \mathbb{N}$. A sequence of *d-regular graphs* $\{G_i\}_{i \in \mathbb{N}}$ of *size increasing with i* is a family of expander graphs if there exists $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$ for all i.

Graphs as Matrices

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The adjacency matrix of an n-vertex graph G, denoted $A = A(G)$, is an $n \times n$ matrix $(a_{i,j})_{i,j \in [n]}$ where

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Theorem

If A is real and symmetric, then

- *A has exactly n (not necessarily distinct) eigenvalues, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.*
- *There exists a set of n eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, one for each eigenvalue (i.e., $Av_i = \lambda_i v_i$), that are orthonormal, i.e., all v_is are of length 1 and orthogonal with each other.*

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  Note, $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$.

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Theorem

Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

Note $d - \lambda_2 = \lambda_1 - \lambda_2$ is the spectral gap of G.

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Lemma

Let $G = (V, E)$ be a d-regular graph with n-vertices. Then for all $S, T \subseteq V$,

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda\sqrt{|S||T|}.$$

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Proof.

Let λ be any eigenvalue of $A = A(G) = (a_{i,j})_{i,j \in [n]}$ with corresponding eigenvector $v = (e_1, \ldots, e_n)$. We choose an $i \in [n]$ such that for every $j \in [n]$

$$|e_i| \geq |e_j|.$$

By $Av = \lambda v$,

$$\sum_{k \in [n]} a_{i,k} e_k = \lambda e_i,$$

Then,

$$d|e_i| = |e_i| \sum_{k \in [n]} |a_{i,k}| \geq \sum_{k \in [n]} |a_{i,k}||e_k| \geq \left| \sum_{k \in [n]} a_{i,k} e_k \right| = |\lambda||e_i|.$$

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Without loss of generality, we assume $e_i > 0$.

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where $\{i, k\} \in E$ means that *there is at least one edge between the i-th and the k-th vertices*.

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Let $\alpha < 1/2$. Then all (n, d, α) graphs are (c, d)-good for some appropriate $c \in \mathbb{N}$.

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Then for some $r = O(\log n)$ and every $x \in V$ we have $|B(x,r)| > n/2$. □

Random Walks on Expander Graphs

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- $\hat{A}$ s real, symmetric, and doubly stochastic; i.e. every column and every row sums up to 1.

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- $\hat{A}$ s real, symmetric, and doubly stochastic; i.e. every column and every row sums up to 1.
- If $d = \lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of A, then $1 = \hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_n$ with $\hat{\lambda}_i = \lambda_i / d$ for $i \in [n]$ are the eigenvalues of $\hat{A}$ with the same eigenvectors as of A.

Normalized Adjacency Matrices

Let G be a d-regular graph with adjacency matrix $A = A(G)$. Then its normalized adjacency matrix is

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- Sampling a vertex x from some probability distribution p on V and then moving to a random neighbor of x is equivalent to sampling a vertex from the distribution $\hat{A}p$.

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- The *stationary distribution* of the random walk on G is the uniform distribution, namely, $u\hat{A} = \hat{A}u = u$.

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Rapid Mixing of Walks

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Theorem

Let G be an (n, d, α)-graph with normalized adjacency matrix $\hat{A}$. Then for any distribution vector p and any positive integer t:

$$\|\hat{A}^t p - u\|_1 \leq \sqrt{n}\,\alpha^t$$

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Corollary

(n, d, α)-graphs are (c, d)-good with

$$c = -\frac{3}{2\log\alpha} + \varepsilon.$$

and $\varepsilon > 0$.

Rapid Mixing of Walks (cont'd)

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Theorem

Let G be an (n, d, α)-graph with normalized adjacency matrix $\hat{A}$. Then for any distribution vector p and any positive integer t:

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Lemma

For every probability vector p, $\|\hat{A}p - u\|_2 \leq \|p - u\|_2 \alpha \leq \alpha$.

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$$\begin{aligned}
\|\hat{A}p - u\|_2 &= \|\hat{A}p - \hat{A}u\|_2 = \|\hat{A}(p-u)\|_2 \\
&= \left\| \sum_{2\le i\le n} \hat{\lambda}_i \beta_i v_i \right\|_2 = \sqrt{\left\langle \sum_{2\le i\le n} \hat{\lambda}_i \beta_i v_i, \sum_{2\le i\le n} \hat{\lambda}_i \beta_i v_i \right\rangle} \\
&= \sqrt{\sum_{2\le i\le n} \hat{\lambda}_i^2 \beta_i^2 \langle v_i, v_i\rangle} \le \alpha \sqrt{\sum_{2\le i\le n} \beta_i^2 \langle v_i, v_i\rangle} \\
&= \alpha \left\| \sum_{2\le i\le n} \beta_i v_i \right\|_2 = \alpha \|p-u\|_2 \le \alpha.
\end{aligned}$$

□