Expander Graphs and Their Applications (IV)

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Last Lecture

Rapid Mixing of Walks

Lemma

For every probability vector p, $\|\hat{A}p - u\|_2 \leq \|p - u\|_2 \alpha \leq \alpha$.

Theorem

Let G be an (n, d, α)-graph with normalized adjacency matrix $\hat{A}$. Then for any distribution vector p and any positive integer t:

$$\|\hat{A}^t p - u\|_2 \leq \|p - u\|_2 \alpha^t \leq \alpha^t.$$

Sampling by Random Walks on Expander Graphs

Let $G = (V, E)$ be an (n, d, α)-graph, and $B \subseteq V$ with $|B| = \beta n$.

We carry out the following experiment: We pick $X_0 \in V$ uniformly at random and start from it a random walk $X_0, \ldots, X_t$ on G.

(B, t): this random walk is confined to B, i.e. that $X_i \in B$ for all $i \in [t]$.

Theorem (**Ajtai**, **Komlós**, and **Szemerédi**, 1987)

Let $G = (V, E)$ be an (n, d, α)-graph and $B \subseteq V$ with $B = \beta n$. Then

$$\Pr\left[(B, t)\right] \leq (\beta + \alpha)^t.$$

Expansion and Spectral Gap

Recall:

Theorem (**Dodziuk**, 84; **Alon** and **Milman**, 85; **Alon**, 86)

Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}.$$

$d - \lambda_2 = \lambda_1 - \lambda_2$ is the spectral gap of G.

Another view of λ

Another view of λ

Lemma

$$\lambda = \max \left\{ \frac{\|Av\|_2}{\|v\|_2} \;\middle|\; v \textit{ and } \mathbf{1} \textit{ are orthogonal, i.e., } \langle v, \mathbf{1} \rangle = 0 \right\}.$$

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Proof.

By $\langle v, \mathbf{1} \rangle = 0$,

$$v = \sum_{2 \le i \le n} \beta_i v_i.$$

Then

$$\|Av\|_2 = \left\| \sum_{2 \le i \le n} \lambda_i \beta_i v_i \right\|_2 \le \lambda \left\| \sum_{2 \le i \le n} \beta_i v_i \right\|_2 = \lambda \|v\|_2.$$

The equality is attained when $v = v_2$ or $v = v_n$. □

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Lemma

$$\lambda_2 = \max\left\{ \frac{vAv}{\|v\|_2^2} \,\middle|\, v \textit{ and } \mathbf{1} \textit{ are orthogonal, i.e., } \langle v, \mathbf{1} \rangle = 0 \right\}.$$

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Proof.

By $\langle v, \mathbf{1}\rangle = 0$,

$$v = \sum_{2 \le i \le n} \beta_i v_i.$$

Then

$$vAv = \langle v, Av\rangle = \left\langle \sum_{2\le i\le n} \beta_i v_i, \sum_{2\le i\le n} \lambda_i \beta_i v_i \right\rangle$$

$$= \sum_{2\le i\le n} \lambda_i \langle \beta_i v_i, \beta_i v_i\rangle \le \lambda_2 \sum_{2\le i\le n} \langle \beta_i v_i, \beta_i v_i\rangle = \lambda_2 \|v\|_2^2.$$

The equality is attained when $v = v_2$. □

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$$\|f\|_2^2 = |\bar{S}|^2 |S| + |S|^2 |\bar{S}| = n|S||\bar{S}|,$$
$$fAf = E(S,S)|\bar{S}|^2 + |E(\bar{S},\bar{S})||S|^2 - 2|S||\bar{S}||E(S,\bar{S})|.$$

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Moreover,

$$E(S,S) = d|S| - |E(S,\bar{S})|, \quad \text{and} \quad E(\bar{S},\bar{S}) = d|\bar{S}| - |E(S,\bar{S})|.$$

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As $\langle f, \mathbf{1} \rangle = 0$,

$$\lambda_2 \geq \frac{fAf}{\|f\|_2^2} = \frac{nd|S||\bar{S}| - n^2|E(S,\bar{S})|}{n|S||\bar{S}|} = d - \frac{n|E(S,\bar{S})|}{|S||\bar{S}|} \geq d - 2h(G).$$

$\square$

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We fix an *arbitrary orientation* of G. Then $K := \underline{K_G} = (k_{v,e})_{v \in V, e \in E}$ is a $|V| \times |E|$-matrix with

$$k_{v,e} := \begin{cases} 1, & v \text{ is the tail of } e, \\ -1, & v \text{ is the head of } e, \\ 0, & \text{otherwise (including } e \text{ is a selfloop on } v). \end{cases}$$

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Lemma

- *$L = KK^T$.*
- *For every $f = (f_v)_{v \in V}$*

$$fLf^T = fKK^Tf^T = \|fK\|_2^2 = \sum_{e \text{ with tail } u \text{ and head } v} (f_u - f_v)^2.$$

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$$fLf^T/\|f\|_2^2 \leq d - \lambda_2 \quad \text{and} \quad h(G)^2/2d \leq fLf^T/\|f\|_2^2.$$

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$fLf^T/\|f\|_2^2 \le d - \lambda_2$: For every $x \in V^+$

$$\begin{aligned}(Lf)_x &= df_x - \sum_{y \in V} a_{x,y} f_y = dg_x - \sum_{y \in V^+} a_{x,y} g_y \\ &\le dg_x - \sum_{y \in V} a_{x,y} g_y = (Lg)_x = (d - \lambda_2) g_x.\end{aligned}$$

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As $f_x = 0$ for $x \notin V^+$ we have

$$fLf^T = \sum_{x \in V} f_x (Lf)_x \le (d-\lambda_2) \sum_{x \in V^+} g_x^2 = (d-\lambda_2) \sum_{x \in V} f_x^2 = (d-\lambda_2)\|f\|_2^2.$$

⊣

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and we will show in the following two lemmas.

$$h(G)\|f\|_2^2 \leq B_f \leq \sqrt{2d}\|fK\|_2\|f\|_2.$$

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Recall: $fLf^T = \|fK\|_2^2$.

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Let $E' := E \setminus \{e \mid e \text{ a selfloop}\}$.

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$$\sqrt{\sum_{E'} \left(f_x + f_y\right)^2} \leq \sqrt{2\sum_{E'} \left(f_x^2 + f_y^2\right)} \leq \sqrt{2d\sum_{V} f_x^2} = \sqrt{2d}\|f\|_2.$$

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Proof.

Reorder the vertices in such a way that *if $x < y$, then $f_x \geq f_y$*.

$$B_f = \sum_{\substack{e \text{ is an edge with} \\ \text{endvertices } x<y}} f_x^2 - f_y^2 = \sum_{\dots} \sum_{i=x}^{y-1} (f_i^2 - f_{i+1}^2) \quad (\text{i.e., } x \leq i \text{ and } y \geq i+1)$$

$$= \sum_{i=1}^{n-1} (f_i^2 - f_{i+1}^2) \left| E([i], [\bar{i}]) \right| = \sum_{i \in V^+} (f_i^2 - f_{i+1}^2) \left| E([i], [\bar{i}]) \right|$$

$$\geq h(G) \sum_{i \in V^+} (f_i^2 - f_{i+1}^2) i \qquad (\text{by } |V^+| \leq |V|/2)$$

$$= h(G) \sum_{i \in V^+} f_i^2 = h(G)\|f\|_2^2.$$

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No! Bipartite graphs. Consequently, we cannot apply, say, Expander Mixing Lemma etc, directly on (combinatorial) expander graphs!

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- $\tilde{G}$ is not bipartite.
- $h(\tilde{G}) = h(G)$.
- $A(\tilde{G}) = A(G) + I$.
- The eigenvalues of $A(\tilde{G})$ are $d+1, \lambda_2 + 1, \ldots, \lambda_n + 1$ with the same corresponding eigenvectors $v_1, v_2, \ldots, v_n$.

G and $\tilde{G}$ (cont'd)

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Theorem

Let $\{G_i\}_{i \in \mathbb{N}}$ be a family of d-regular expander graphs. Then there exists a constant $\delta < d + 1$ such that $\lambda(\tilde{G}_i) \leq \delta$ for every $i \in \mathbb{N}$.

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Bigger expansion of G implies smaller λ of $\tilde{G}$.

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We can prove the above corollary directly using the definition of $h(G)$.