Expander Graphs and Their Applications (V)

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λ, λ_2 and $h(G)$

Recall: $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$

Theorem (**Dodziuk**, 84; **Alon** and **Milman**, 85; **Alon**, 86)

Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

Corollary

$(d - \lambda)/2 \leq h(G)$. *That is, smaller λ implies bigger expansion.*

Does bigger expansion imply smaller λ?

No! Bipartite graphs. Consequently, we cannot apply, say, Expander Mixing Lemma etc, directly on (combinatorial) expander graphs!

G and $\tilde{G}$

Let $G = (V, E)$ be a d-regular graph. We define a $(d+1)$-regular graph $\tilde{G}$ by *adding a (new) selfloop on every vertex in V*.

- $\tilde{G}$ is not bipartite.
- $h(\tilde{G}) = h(G)$.
- $A(\tilde{G}) = A(G) + I$.
- The eigenvalues of $A(\tilde{G})$ are $d+1, \lambda_2 + 1, \ldots, \lambda_n + 1$ with the same corresponding eigenvectors $v_1, v_2, \ldots, v_n$.

G and $\tilde{G}$ (cont'd)

Theorem

Let $\{G_i\}_{i\in\mathbb{N}}$ be a family of d-regular expander graphs. Then there exists a constant $\delta < d+1$ such that $\lambda(\tilde{G}_i) \leq \delta$ for every $i \in \mathbb{N}$.

Proof.

As $h(\tilde{G}_i) = h(G_i) \geq \varepsilon$ for a fixed constant $\varepsilon > 0$, by **Dodziuk**, **Alon** et.al. we have

$$\lambda_2(\tilde{G}_i) \leq \iota.$$

for a fixed constant $\iota < d+1$.

If $\lambda(\tilde{G}_i) = \lambda_2(\tilde{G}_i)$, then $\lambda(\tilde{G}_i) \leq \iota$.

If $\lambda(\tilde{G}_i) = -\lambda_n(\tilde{G}_i)$, then $\lambda(\tilde{G}_i) = -(\lambda_n(G_i)+1) \leq d-1$.

So we can take $\delta := \max\{\iota, d-1\}$. □

Bigger expansion of G implies smaller λ of $\tilde{G}$.

Expander Graphs are Good

Recall:

Corollary

For every $\varepsilon > 0$*,* (n, d, α)*-graphs are* (c, d)*-good with* $c = -3/(2\log\alpha) + \varepsilon$*.*

Corollary

Let $\{G_i\}_{i\in\mathbb{N}}$ *be a family of* d*-regular expander graphs. Then* G_i*s are* (c, d)*-good for some appropriate* $c \in \mathbb{N}$*.*

Proof.

$\tilde{G}_i$ is a $(n, d+1, \alpha)$-graph with a fixed constant $\alpha < 1$.

$\tilde{G}_i$ is $(c, d+1)$-good for some fixed $c \in \mathbb{N}$.

G_i is (c, d)-good. □

We can prove the above corollary directly using the definition of $h(G)$*.*

The Existence of Expander Graphs

Using *probabilistic method* we can prove that

almost every graph is an expander graph.

In particular:

Theorem

For every sufficient large $d \in \mathbb{N}$, there exists a $(d^4, d, 1/4)$*-graph.*

The Inductive Construction of Expander Graphs

The (Normal) Graph Product

Let G be a graph with adjacency matrix A and $k \in \mathbb{N}$, then G^2 is a graph with adjacency matrix A^2. That is, for every pair of vertices u and v

there is an edge between u and v in $G^k \iff$ there is a walk of length k in G.

Theorem

If G is an (n, d, α)-graph, then G^k is an (n, d^k, α^k)-graph.

Inductive Construction Using the Zig-Zag Product

Let G and H be two graphs. We will define their zig-zag product $G \textcircled{z} H$ in such a way that the following theorem is true.

Theorem (**Reingold**, **Vadhan**, and **Wigderson**, 02)

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G \textcircled{z} H$ is an $(nm, d^2, \varphi(\alpha, \beta))$-graph, where the function φ satisfies the following conditions:

- *If $\alpha < 1$ and $\beta < 1$, then $\varphi(\alpha, \beta) < 1$.*
- $\varphi(\alpha, \beta) \leq \alpha + \beta$.
- $\varphi(\alpha, \beta) \leq 1 - (1 - \beta^2)(1 - \alpha)/2$.

Inductive Construction Using Zig-Zag Products (cont'd)

Let H be a $(d^4, d, 1/4)$-graph for a constant d. Then we let

$$G_1 := H^2, \quad \text{and} \quad G_{n+1} := G_n^2 \circledz H \text{ for } n \geq 1.$$

Theorem

Every graph G_n is a $(d^{4n}, d^2, 1/2)$-graph for all $n \in \mathbb{N}$.

Proof.

Trivial for G_1 by the normal graph product.
By induction hypothesis, G_n^2 is a $(d^{4n}, d^4, 1/4)$-graph. Then by the Zig-Zag Theorem,

$$\lambda(G_{n+1}) \leq \lambda(G_n^2) + \lambda(H) = 1/4 + 1/4 = 1/2.$$

□

The Definition of the Zig-Zag Product

Let G be an (n, m, α)-graph and H an (m, d, β)-graph. For every vertex $v \in V(G)$ we fix some numbering

$$e_v^1, e_v^2, \ldots, e_v^m$$

of the edges incident to v in G. We also assume

$$V(H) = [m] = \{1, 2, \ldots, m\}.$$

Definition

$G \textcircled{z} H = (V(G) \times [m], E')$ where

$$\begin{aligned} & \text{there is an edge between } (v,i) \text{ and } (u,j) \\ \Longleftrightarrow \quad & \text{there exist some } k, \ell \in [m] \text{ such that} \\ & \quad \text{in } H \text{ there are edges between } i \text{ and } k,\ \ell \text{ and } j \\ & \quad \text{and } e_v^k = e_u^\ell \text{ in } G. \end{aligned}$$

The Adjacency Matrix of the Zig-Zag Product

Every edge between (v, i) and (u, j) in $G ⓩ H$ can be viewed a walk length 3 consisting of three steps:

(i) an edge in the v-th copy in H, i.e., the edge between i and k;

(ii) an edge in G, i.e., the edge $e_v^k = e_u^\ell$;

(iii) an edge in the u-th copy of H, i.e., the edge between ℓ and j.

Let $\hat{B}$ be the normalized adjacency matrix of H. (i) and (iii) are done on n disjoint copies of H with the corresponding transition matrix

$$\tilde{B} := \hat{B} \otimes I_n \qquad \textit{(tensor product)}$$

The Adjacency Matrix of the Zig-Zag Product (cont'd)

In step (ii) we move from a vertex (v, k) to the *unique* vertex (u, ℓ) with $e_v^k = e_u^\ell$. Thus, the corresponding transition matrix P is defined by

$$P_{(v,k),(u,\ell)} := \begin{cases} 1, & \text{if } e_v^k = e_u^\ell \\ 0, & \text{otherwise.} \end{cases}$$

P is a *permutation matrix*, hence for every f

$$\|Pf\|_2 = \|f\|_2.$$

In total, the normalized adjacency matrix of $G \textcircled{z} H$ is

$$Z := \tilde{B} P \tilde{B}.$$

A Weak Version of the Zig-Zag Theorem

Theorem

If G is an (n, m, α)-graph and H an (m, d, β)-graph. Then $G \circledz H$ is an (nm, d^2, φ)-graph with

$$\varphi \leq \alpha + \beta + \beta^2.$$

Yet another view of λ

Recall we have proved
$\lambda_2 = \max\left\{ vAv/\|v\|_2^2 \middle| \ v \text{ and } \mathbf{1} \text{ are orthogonal, i.e., } \langle v, \mathbf{1}\rangle = 0\right\}.$

Lemma

$$\lambda = \max\left\{ \frac{|vAv|}{\|v\|_2^2} \middle| \ v \textit{ and } \mathbf{1} \textit{ are orthogonal, i.e., } \langle v, \mathbf{1}\rangle = 0\right\}.$$

Proof.

By $\langle v, \mathbf{1}\rangle = 0$, $v = \sum_{2\le i\le n} \beta_i v_i$. Then

$$|vAv| = |\langle v, Av\rangle| = \left|\left\langle \sum_{2\le i\le n} \beta_i v_i, \sum_{2\le i\le n} \lambda_i \beta_i v_i \right\rangle\right|$$

$$= \left|\sum_{2\le i\le n} \lambda_i \langle \beta_i v_i, \beta_i v_i\rangle\right| \le \lambda \sum_{2\le i\le n} \langle \beta_i v_i, \beta_i v_i\rangle = \lambda \|v\|_2^2.$$

The equality is attained when $v = v_2$ *or* $v = v_n$. □

Proof

By the previous lemma, it suffices to show that for every mn-vector f with $\langle f, \mathbf{1}\rangle = 0$ we have

$$\frac{|fZf|}{\|f\|_2^2} \leq \alpha + \beta + \beta^2$$

Define $\underline{f^{\|}}$ by

$$f^{\|}(x,i) := \frac{\sum_{j\in[m]} f(x,j)}{m}$$

and

$$\underline{f^{\perp}} := f - f^{\|}.$$

It is easy to verify that $\left\langle f^{\|}, f^{\perp}\right\rangle = 0$, hence

$$\|f\|_2^2 = \|f^{\|}\|_2^2 + \|f^{\perp}\|_2^2.$$

Proof (cont'd)

$$\left\|fZf\right\| = \left|f\tilde{B}P\tilde{B}f\right| \leq \left|f^{\|}\tilde{B}P\tilde{B}f^{\|}\right| + 2\left|f^{\|}\tilde{B}P\tilde{B}f^{\perp}\right| + \left|f^{\perp}\tilde{B}P\tilde{B}f^{\perp}\right|$$

- $\tilde{B}f^{\|} = f^{\|}$ by $\hat{B}\mathbf{1}_m = \mathbf{1}_m$.
- $\left\|\tilde{B}f^{\perp}\right\|_2 \leq \beta\|f^{\perp}\|_2$ by $\|\hat{B}u\|_2 \leq \beta\|u\|_2$ for every $\langle u, \mathbf{1}\rangle = 0$.

Then

$$\begin{aligned}\|fZf\| &\leq \left|f^{\|}Pf^{\|}\right| + 2\left|f^{\|}P\tilde{B}f^{\perp}\right| + \left|f^{\perp}\tilde{B}P\tilde{B}f^{\perp}\right| \\ &\leq \left|f^{\|}Pf^{\|}\right| + 2\beta\|f^{\|}\|_2\|f^{\perp}\|_2 + \beta^2\|f^{\perp}\|_2^2.\end{aligned}$$

The second inequality uses the fact that P is a permutation and the fact that $|f_1 f_2| \leq \|f_1\|_2\|f_2\|_2$.

Proof (cont'd)

Then we show

$$\left| f^{\parallel} P f^{\parallel} \right| \leq \alpha \| f^{\parallel} \|_2^2.$$

Define a function/vector g on $V(G)$ by

$$g(v) := \sqrt{m} f^{\parallel}(v, i) = \frac{\sum_{j \in [m]} f(x, j)}{\sqrt{m}}.$$

By **Cauchy-Schwartz**, $\|g\|_2^2 \leq \|f\|_2^2$.
By the definition of P,

$$f^{\parallel} P f^{\parallel} = g \hat{A} g$$

where $\hat{A}$ is the normalized adjacency matrix of G.
As $\left\langle f^{\parallel}, \mathbf{1}_{mn} \right\rangle = 0$, we have $\langle g, \mathbf{1}_n \rangle = 0$. Therefore, $|g \hat{A} g| \leq \alpha \|g\|_2^2$. The result follows.

Proof (cont'd)

Finally we have

$$\|fZf\| \leq \alpha \|f^{\|}\|_2^2 + 2\beta \|f^{\|}\|_2 \|f^{\perp}\|_2 + \beta^2 \|f^{\perp}\|_2^2.$$

Recall $\|f\|_2^2 = \|f^{\|}\|_2^2 + \|f^{\perp}\|_2^2$,

$$\frac{\|fZf\|}{\|f\|_2^2} \leq \alpha + \beta + \beta^2.$$

$\square$