# Expander Graphs and Their Applications (VII)

Yijia Chen
Shanghai Jiaotong University

# Last Lecture

# The Zig-Zag Theorem

## Theorem

*If $G$ is an $(n, m, \alpha)$-graph and $H$ an $(m, d, \beta)$-graph. Then $G \textcircled{z} H$ is an $(nm, d^2, \varphi(\alpha, \beta))$-graph, where*

$$\varphi(\alpha, \beta) = \frac{1}{2}(1 - \beta^2)\alpha + \frac{1}{2}\sqrt{(1 - \beta^2)^2\alpha^2 + 4\beta^2}.$$

Some consequences:

- $\varphi(\lambda, 0) = \varphi(0, \lambda) = \lambda$ and $\varphi(\lambda, 1) = \varphi(1, \lambda) = 1$ for all $0 \leq \lambda \leq 1$.
- $\varphi(\alpha, \beta)$ is a strictly increasing function of both $\alpha$ and $\beta$ (except when one of them is 1).
- If $\alpha < 1$ and $\beta < 1$, the $\varphi(\alpha, \beta) < 1$.
- $\varphi(\alpha, \beta) \leq \alpha + \beta$ for all $0 \leq \alpha, \beta \leq 1$.

## Corollary

*If $G$ is an $(n, m, \alpha)$-graph and $H$ an $(m, d, \beta)$-graph. Then*

$$1 - \hat{\lambda}(G \textcircled{z} H) \geq \frac{1}{2}(1 - \beta^2)(1 - \alpha)$$

## Proof.

$$\frac{1}{2}\sqrt{(1-\beta^2)^2\alpha^2 + 4\beta^2} \leq \frac{1}{2}\sqrt{(1-\beta^2)^2 + 4\beta^2} = \frac{1}{2}(1+\beta^2) = 1 - \frac{1}{2}(1-\beta^2).$$

Then

$$\begin{aligned}\varphi(\alpha, \beta) &= \frac{1}{2}(1-\beta^2)\alpha + \frac{1}{2}\sqrt{(1-\beta^2)^2\alpha^2 + 4\beta^2} \\ &\leq 1 - \frac{1}{2}(1-\beta^2)(1-\alpha)\end{aligned}$$

□

# Lower Bound of $d - \lambda$ in Arbitrary Graphs

### Theorem

*Let $G$ be a connected, $d$-regular, and non-bipartite graph with $n$ vertices. Then*

$$d - \lambda \geq \frac{1}{n^2}.$$

It suffices to show the following two lemmas.

### Lemma

*Let $G$ be a connected and $d$-regular graph with $n$ vertices. Then*

$$d - \lambda_2 \geq \frac{1}{n^2}.$$

### Lemma

*Let $G$ be a connected, $d$-regular, and non-bipartite graph with $n$ vertices. Then*

$$d + \lambda_n \geq \frac{1}{n^2}.$$

# $d - \lambda_2 \geq 1/n^2$

**Lemma**

*Let $G$ be a connected and $d$-regular graph with $n$ vertices. Then* $d - \lambda_2 \geq 1/n^2$.

**Proof.** Recall $\lambda_2 = \max_{\langle x, \mathbf{1} \rangle = 0} xAx/\|x\|_2^2 = \max_{\langle x, \mathbf{1} \rangle = 0, \|x\|_2 = 1} xAx$.

$$\lambda_2 = \max_{\langle x, \mathbf{1} \rangle = 0, \|x\|_2 = 1} \left( \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} 2x_u x_v + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2 \right)$$

$$= \max_{\langle x, \mathbf{1} \rangle = 0, \|x\|_2 = 1} \left( \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} \left(x_u^2 + x_v^2 - (x_u - x_v)^2\right) + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2 \right)$$

$$= \max_{\langle x, \mathbf{1} \rangle = 0, \|x\|_2 = 1} \left( \sum_{u \in V} d x_u^2 - \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \text{ and } v}} (x_u - x_v)^2 \right).$$

# Proof of $d - \lambda_2 \geq 1/n^2$ (cont'd)

Then

$$\lambda_2 = d - \max_{\langle x, \mathbf{1} \rangle = 0, \|x\|_2 = 1} \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \text{ and } v}} (x_u - x_v)^2,$$

i.e.,

$$d - \lambda_2 = \min_{\langle x, \mathbf{1} \rangle = 0, \|x\|_2 = 1} \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \text{ and } v}} (x_u - x_v)^2.$$

Choose some $x$ with $\langle x, \mathbf{1} \rangle = 0$ and $\|x\|_2 = 1$ such that

$$d - \lambda_2 = \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \text{ and } v}} (x_u - x_v)^2.$$

(Here we use the fact that the minimum can be attained.)

# Proof of $d - \lambda_2 \geq 1/n^2$ (cont'd)

Since $\|x\|_2 = 1$, there exists some $u_0 \in V$ such that

$$|x_{u_0}| \geq \frac{1}{\sqrt{n}}.$$

As $\langle x, \mathbf{1} \rangle = 0$, i.e., $\sum_u x_u = 0$, there exists some $u_1 \in V$ such that

$x_{u_0}$ and $x_{u_1}$ have different signs, i.e., $x_{u_0} x_{u_1} < 0$.

Then we conclude

$$|x_{u_0} - x_{u_1}| \geq \frac{1}{\sqrt{n}}.$$

Recall $G$ is connected. Thus, there exists a path

$$u_0 = v_0, v_1, \ldots, v_k = u_1.$$

Hence,

$$\sum_{i=0}^{k-1} |x_{v_i} - x_{v_{i+1}}| \geq |x_{v_0} - x_{v_k}| \geq \frac{1}{\sqrt{n}}.$$

# Proof of $d - \lambda_2 \geq 1/n^2$ (cont'd)

$$\begin{aligned} d - \lambda_2 &= \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \text{ and } v}} (x_u - x_v)^2. \\ &\geq \sum_{i=0}^{k-1} \left(x_{v_i} - x_{v_{i+1}}\right)^2 \\ &\geq \frac{1}{k} \left( \sum_{i=0}^{k-1} \left| x_{v_i} - x_{v_{i+1}} \right| \right)^2 \\ &\geq \frac{1}{k} \left| x_{v_0} - x_{v_k} \right|^2 \\ &\geq \frac{1}{kn} \geq \frac{1}{n^2}. \end{aligned}$$

□

## $d + \lambda_n \geq 1/n^2$

**Lemma**

*Let $G$ be a connected, $d$-regular, and non-bipartite graph with $n$ vertices. Then*

$$d + \lambda_n \geq \frac{1}{n^2}.$$

**Proof.** Let $x$ be an eigenvector for $\lambda_n$ with $\|x\|_2 = 1$.

$$\lambda_n = \lambda_n \|x\|_2^2 = xAx = \sum_{u,v \in [n]} a_{u,v} x_u x_v = \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} 2x_u x_v + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2$$

Thus,

$$\begin{aligned} d + \lambda_n &= \sum_{u \in V} d x_i^2 + \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} 2x_u x_v + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2 \\ &= \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} (x_u + x_v)^2 + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2. \end{aligned}$$

## Proof of $d + \lambda_n \geq 1/n^2$ (cont'd)

We assume:

- $V = [n] = \{1, 2, \ldots, n\}$.
- $x_1$ is positive and has maximum absolute value among all $x_i$.
- Recall that $G$ is *non-bipartite*, so there exist an edge with end vertices $i, j \in [n]$ such that $x_i, x_j \geq 0$ or $x_i, x_j < 0$.

## $x_i, x_j \geq 0$ and $i \neq j$:

As $G$ is connected, without loss of generality, we can assume that

$$1, 2, \ldots, i, i+1 = j$$

is a shortest path from 1 to $i$ and $j$. By choosing either $k = i$ or $k = i+1$ we have a path

$$1, 2, \ldots, k$$

with odd length and $x_k \geq 0$

$$\begin{aligned}
d + \lambda_n &\geq \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} (x_u + x_v)^2 \geq \sum_{i \in [k-1]} (x_i + x_{i+1})^2 \\
&\geq \frac{1}{k-1} \left( \sum_{i \in [k-1]} |x_i + x_{i+1}| \right)^2 \\
&\geq \frac{1}{k-1} \left( (x_1 + x_2) + (-x_2 - x_3) + \ldots + (x_{k-1} + x_k) \right)^2 = \frac{(x_1 + x_k)^2}{k-1} \\
&\geq \frac{x_1^2}{n} \geq \frac{\sum_{i \in [n]} x_i^2}{n^2} = \frac{1}{n^2}.
\end{aligned}$$

## $x_i, x_j \geq 0$ and $i = j$:

Again we can assume that

$$1, 2, \ldots, i$$

is a shortest path from 1 to $i$ and $j$. If the path is of odd length, i.e., $i$ is even, then we are done by the previous proof. Otherwise, it is of even length.

$$\begin{aligned}
d + \lambda_n &= \sum_{\substack{\text{an edge } e \text{ with} \\ \text{end vertices } u \neq v}} (x_u + x_v)^2 + \sum_{\substack{\text{a selfloop } e \\ \text{on an vertex } u}} x_u^2 v \\
&\geq \sum_{j \in [i-1]} (x_j + x_{j+1})^2 + x_i^2 \\
&\geq \frac{1}{i} \left( |x_i| + \sum_{j \in [i-1]} |x_j + x_{j+1}| \right)^2 \\
&\geq \frac{1}{i} \left( (x_1 + x_2) + (-x_2 - x_3) + \ldots + (-x_{i-1} - x_i) + x_i \right)^2 \\
&\geq \frac{x_1^2}{n} \geq \frac{\sum_{i \in [n]} x_i^2}{n^2} = \frac{1}{n^2}.
\end{aligned}$$

## $x_i, x_j < 0$ and $i \neq j$:

Again, we can assume that

$$1, 2, \ldots, i, i+1 = j$$

is a shortest path from 1 to $i$ and $j$. By choosing either $k = i$ or $k = i+1$ we have a path

$$1, 2, \ldots, k$$

with even length and $x_k < 0$

$$\begin{aligned} d + \lambda_n &\geq \frac{1}{k-1}\left(\sum_{i \in [k-1]} |x_i + x_{i+1}|\right)^2 \\ &\geq \frac{1}{k-1}\left((x_1 + x_2) + (-x_2 - x_3) + \ldots + (-x_{k-1} - x_k)\right)^2 = \frac{(x_1 - x_k)^2}{k-1} \\ &\geq \frac{x_1^2}{n} \geq \frac{\sum_{i \in [n]} x_i^2}{n^2} = \frac{1}{n^2}. \end{aligned}$$

## $x_i, x_j < 0$ and $i = j$:

Again, we can assume that

$$1, 2, \ldots, i$$

is a shortest path from 1 to $i$ and $j$. If the path is of even length then we are done by the previous proof. Otherwise it is of odd length.

$$\begin{aligned} d + \lambda_n &\geq \frac{1}{i}\left(|x_i| + \sum_{j \in [i-1]} |x_j + x_{j+1}|\right)^2 \\ &\geq \frac{1}{i}\left((x_1 + x_2) + (-x_2 - x_3) + \ldots + (x_{i-1} + x_i) - x_i\right)^2 \\ &\geq \frac{x_1^2}{i} \geq \frac{\sum_{i \in [n]} x_i^2}{n^2} = \frac{1}{n^2}. \end{aligned}$$

□

# Reingold's Algorithm

# Graph Reachability Problem

| | |
|---|---|
| *Input:* | A graph $G = (V, E)$ and $u, v \in V$. |
| *Problem:* | Is there a path in $G$ from $u$ to $v$? |

**Theorem (Omer Reingold, 2005)**

*There is an algorithm that solves the reachability problem on any graph $G = (V, E)$ using space $O(\log |V|)$.*

An important fact: If we design an algorithm by merging several subprograms, each using logspace, then the whole program can use only logspace.

*For a proof, see* **Papadimitriou**'*s Computational Complexity.*

# Every graph can be made into degree 3

From now on we always assume every graph $H = (V(H), E(H))$ has no *vertex of degree at most 2*.

For every vertex $v \in V(H)$, assume $v$ is of degree $m$, then we fix some numbering

$$e_v^1, e_v^2, \ldots, e_v^m$$

of the edges incident to $v$ in $H$.

We construct the following graph $\underline{G^{H,3}}$ where

$$V(G^{H,3}) := \{(v, i) \mid v \in V(H) \text{ and } 1 \leq i \leq \deg(v) \text{ (the degree of } v)\}.$$

and for the edge set $E(H^{G,3})$

- for every vertex $v \in V(H)$ and $i \in [\deg(v)]$, there is en edge between $(v, i)$ and $(v, (i+1) \mod \deg(v))$;
- for every $u, v \in V(H)$ with $u \neq v$ and $i, j \in \mathbb{N}$, there is an edge between $(u, i)$ and $(v, j)$ if $e_u^i = e_v^j$.

# Every graph can be made into degree 3 (cont'd)

Theorem

- *$G^{H,3}$ is 3-regular or empty.*
- *For every $u, v \in V(H)$*

$$\begin{aligned} & \text{there is a path between } u \text{ and } v \text{ in } H \\ \iff \quad & \text{there is a path between } (u,1) \text{ and } (v,1) \text{ in } G^{H,3}. \end{aligned}$$

# Every graph can be made into constant degree

Let $H$ be a graph and $D \geq 3$. We construct the graph $G^{H,D}$ by *adding* $(D-3)$ *many selfloops to each vertex in* $G^{H,3}$.

Theorem

- $G^{H,D}$ *is D-regular or empty.*
- *For every* $u, v \in V$

$$\begin{aligned} & \text{there is a path between } u \text{ and } v \text{ in } H \\ \iff \quad & \text{there is a path between } (u,1) \text{ and } (v,1) \text{ in } G^{H,D}. \end{aligned}$$

# Reingold's Algorithm

Let $H$ be a $(d^{16}, d, 1/2)$-graph for some constant $d \in \mathbb{N}$.

**Input:** A $d^{16}$-regular graph $G$ with $n := |V(G)|$ and two vertices $s, t \in V(G)$.
**Output:** Is there a path between $s$ and $t$ in $G$?

1. Choose some $\ell = O(\log n)$ such that

$$\left(1 - \frac{1}{d^{16} n^2}\right)^{2^\ell} \leq \frac{1}{2}.$$

2. $G_0 \leftarrow G$.
3. **for** $i = 1$ **to** $\ell$ **do** $G_i \leftarrow (G_{i-1} ⓩ H)^8$.
4. Check if $s$ and $t$ are connected in $G_\ell$.

# The Analysis of Reingold's Algorithm

**Proposition** *For all $i \in [\ell]$*

$$\hat{\lambda}(G_i) \le \max\left\{\hat{\lambda}^2(G_{i-1}), \frac{1}{2}\right\}.$$

**Proof.**

Recall for every $G$ and $H$ we have proved $1-\hat{\lambda}(G \circledz H) \ge (1-\beta^2)(1-\alpha)/2$. Now our $H$ is a $(d^{16}, d, 1/2)$-graph, hence

$$\hat{\lambda}(G_{i-1} \circledz H) \le 1 - 3\big(1-\hat{\lambda}(G_{i-1})\big)/8 \le 1 - \big(1-\hat{\lambda}(G_{i-1})\big)/3.$$

Case 1. $\hat{\lambda}(G_{i-1}) \le 1/2$. Then

$$\hat{\lambda}(G_i) = \Big(\hat{\lambda}(G_{i-1} \circledz H)\Big)^8 \le (1-1/6)^8 < 1/2.$$

Case 2. $\hat{\lambda}(G_{i-1}) > 1/2$. By *expansion*, for all $1/2 \le x \le 1$ we have $(1-(1-x)/3)^4 \le x$. Then

$$\hat{\lambda}(G_i) \le (1-(1-\hat{\lambda}(G_{i-1}))/3)^8 \le \hat{\lambda}^2(G_{i-1}).$$

□

# The Analysis of Reingold's Algorithm (cont'd)

For all $i \in [\ell]$ and every *connected component* $C$ of $G$, we define

$$C_0 := C \quad \text{and} \quad C_i := (C_{i-1} \circled{z} H)^8$$

It is easy to see

$$G_i := \bigcup_{\substack{C \text{ a connected} \\ \text{component of } G}} C_i$$

**Proposition**

$$\hat{\lambda}(C_i) \leq \max\left\{\hat{\lambda}^2(C_{i-1}), \frac{1}{2}\right\}.$$

# The Analysis of Reingold's Algorithm (cont'd)

### Corollary

*For every connected component $C$ of $G$*

$$\hat{\lambda}(C_\ell) \leq \frac{1}{2}.$$

### Proof.

Recall

- Let $G$ be a connected, $d$-regular, and non-bipartite graph with $n$ vertices. Then $d - \lambda \geq 1/n^2$.
- We choose $\ell$ in such a way that

$$\left(1 - \frac{1}{d^{16} n^2}\right)^{2^\ell} \leq \frac{1}{2}.$$

□