Mathematical Logic (XI)

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1. Decidability and Enumerability

1.1. The undecidability of first-order logic. We will make use of the following theorem.

Theorem 1.1. *Let $\mathcal{A}$ be a fixed alphabet.*

(i) The set

$$\Pi'_{\mathrm{halt}} := \left\{ w_{\mathbb{P}} \;\middle|\; \mathbb{P} \text{ a program over } \mathcal{A} \text{ and } \mathbb{P} : w_{\mathbb{P}} \to \mathrm{halt} \right\}$$

is not R-decidable.

(ii) The set

$$\Pi_{\mathrm{halt}} := \left\{ w_{\mathbb{P}} \;\middle|\; \mathbb{P} \text{ a program over } \mathcal{A} \text{ and } \mathbb{P} : \square \to \mathrm{halt} \right\}$$

is not R-decidable. ⊣

Theorem 1.2. *The set*

$$\left\{ \varphi \in L_0^{S_\infty} \;\middle|\; \models \varphi \right\} \tag{1}$$

is not R-decidable.

Proof: By Theorem 1.1 (ii) for the alphabet $\mathcal{A} = \{|\}$ the problem Π_{halt} is not R-decidable. Our goal is to show that the assumed R-decidability of (1) would contradict this result. To that end, for every program $\mathbb{P}$ we will construct in an *effective* way a $\varphi_{\mathbb{P}} \in L_0^{S_\infty}$ such that

$$\mathbb{P} : \square \to \mathrm{halt} \quad \iff \quad \models \varphi_{\mathbb{P}}.$$

Here, the effectivity means that there is a program $\mathbb{P}_1$ which computes the mapping

$$\mathbb{P} \mapsto \varphi_{\mathbb{P}}.$$

Once this is done, given an input $w \in \mathcal{A}^*$, we can first check whether $w = w_{\mathbb{P}}$, if so, extract the program $\mathbb{P}$ and compute $\varphi_{\mathbb{P}}$ using $\mathbb{P}_1$. Thus if (1) is decidable, we can apply the corresponding decision program on input $\varphi_{\mathbb{P}}$ to decide whether $\mathbb{P} : \square \to \mathrm{halt}$. Hence, we could decide Π_{halt}.

Let $\mathbb{P}$ consist of instructions $\alpha_0, \ldots, \alpha_k$, in particular every α_i has its label i. Furthermore, assume that the maximum index of the registers which $\mathbb{P}$ uses is n. Hence, the registers referred by all α_i's are among $R_0, \ldots, R_n$.

Key to our construction of $\varphi_{\mathbb{P}}$ is the notion of configurations of $\mathbb{P}$. A $(n+2)$-tuple

$$(L, m_0, \ldots, m_n)$$

is a *configuration of* $\mathbb{P}$ *(on input* $\square$*) after* s *steps* if

- starting with input $\square$ the program $\mathbb{P}$ runs at least s steps,
- after s steps, the instruction α_L is to be executed next,

– and for every $0 \leqslant i \leqslant n$ the register R_i contains the word

$$\underbrace{||\cdots|}_{m_i \text{ times}}$$

at that moment. To ease presentation, in the following we will simply say that R_i contains the number m_i.

Observe that then the execution of $\mathbb{P}$ on the $s+1$-th step is completely determined by the configuration $(L, m_0, \dots, m_n)$.

The *initial configuration*, i.e., the configuration of $\mathbb{P}$ after 0 step is

$$(0, 0, \dots, 0).$$

Recall that α_k is the last instruction of $\mathbb{P}$, i.e., the only halt instruction. Therefore

$$\mathbb{P} : \square \to \text{halt} \iff \text{for some } s, m_0, \dots, m_n \in \mathbb{N} \\ \text{the tuple } (k, m_0, \dots, m_n) \text{ is the configuration of } \mathbb{P} \text{ after } s \text{ steps.} \quad (2)$$

In case $\mathbb{P} : \square \to \text{halt}$, we define $s_{\mathbb{P}} \in \mathbb{N}$ to be the number of steps which $\mathbb{P}$ carries out until it reaches the last halt instruction.

We choose four symbols from S^∞: $R := R_0^{n+3}$, $< := R_0^2$, $f := f_0^1$, and $c := c_0$, and set

$$S := \{R, <, f, c\}.$$

Then we associate with $\mathbb{P}$ an S-structure $\mathfrak{A}_{\mathbb{P}}$ which "describes" the execution (i.e., the behaviour) of $\mathbb{P}$ on input $\square$. There are two cases.

Case 1. $\mathbb{P} : \square \to \infty$. We set $A_{\mathbb{P}} := \mathbb{N}$, $<^{\mathfrak{A}_{\mathbb{P}}} := \big\{(i, j) \bigm| i, j \in \mathbb{N} \text{ and } i < j\big\}$, $f^{\mathfrak{A}_{\mathbb{P}}}(i) := i + 1$ for every $i \in \mathbb{N}$, $c^{\mathfrak{A}_{\mathbb{P}}} := 0$, and

$$R^{\mathfrak{A}_{\mathbb{P}}} := \big\{(s, L, m_0, \dots, m_n) \bigm| (L, m_0, \dots, m_n) \text{ is the configuration of } \mathbb{P} \text{ after } s \text{ steps}\big\}.$$

Case 2. $\mathbb{P} : \square \to \text{halt}$. Let $e := \max\{k, s_{\mathbb{P}}\}$. Then we set $A_{\mathbb{P}} := \{0, \dots, e\}$, $<^{\mathfrak{A}_{\mathbb{P}}} := \big\{(i, j) \bigm| 0 \leqslant i < j \leqslant e\big\}$, $f^{\mathfrak{A}_{\mathbb{P}}}(i) := \max\{i + 1, e\}$ for every $i \in A_{\mathbb{P}}$, $c^{\mathfrak{A}_{\mathbb{P}}} := 0$, and

$$R^{\mathfrak{A}_{\mathbb{P}}} := \big\{(s, L, m_0, \dots, m_n) \bigm| (L, m_0, \dots, m_n) \text{ is the configuration of } \mathbb{P} \text{ after } s \text{ steps}\big\}.$$

Note that, since every register R_i starts with 0, and can increase its value (i.e, the length of $||\cdots|$) by at most 1 in each step, thus $m_i \leqslant s_{\mathbb{P}} \leqslant e$. So $R^{\mathfrak{A}_{\mathbb{P}}}$ is well defined.

Towards the definition of $\varphi_{\mathbb{P}}$ in (1), we first construct a sentence $\psi_{\mathbb{P}}$ which expresses the execution of $\mathbb{P}$ on $\square$. We abbreviate c, fc, ffc, ... by $\bar{0}$, $\bar{1}$, $\bar{2}$, ..., respectively. The desired $\psi_{\mathbb{P}}$ should satisfy the following two properties:

(P1) $\mathfrak{A}_{\mathbb{P}} \models \psi_{\mathbb{P}}$.

(P2) Let $\mathfrak{A}$ be an S-structure with $\mathfrak{A} \models \psi_{\mathbb{P}}$. Furthermore, $(L, m_0, \dots, m_n)$ is the configuration of $\mathbb{P}$ after s steps. Then

$$\mathfrak{A} \models R\bar{s}\bar{L}\bar{m}_0 \cdots \bar{m}_n.$$

We set

$$\psi_{\mathbb{P}} := \psi_0 \land R\bar{0}\bar{0}\cdots\bar{0} \land \psi_{\alpha_0} \land \cdots \land \psi_{\alpha_{k-1}},$$

where each conjunct is defined as follows. The first

$$\psi_0 := \text{"$<$ is an ordering"} \land \forall x(c < x \lor x \equiv c) \land \forall x(x < fx \lor x \equiv fx) \\ \land \forall x\big(\exists y\, x < y \to (x < fx \land \forall z(x < z \to (fx < z \lor fx \equiv z)))\big),$$

i.e., $<$ is an ordering, c is the minimum element, fx is the successor of x except that $x = fx$ for the maximum x.

For $\alpha \in \{\alpha_0, \dots, \alpha_{k-1}\}$ we define φ_α by a case analysis.

- $\alpha = L$ **LET** $R_i = R_i + |$. Then let
$$\psi_\alpha := \forall x \forall y_0 \cdots \forall y_n \big(Rx\bar{L}y_0 \cdots y_n \to (x < fx \wedge Rfx\overline{L+1}y_0 \cdots y_{i-1}fy_iy_{i+1} \cdots y_n)\big).$$
- $\alpha = L$ **LET** $R_i = R_i - |$. Then let
$$\begin{aligned}\psi_\alpha := \forall x \forall y_0 \cdots \forall y_n \big(Rx\bar{L}y_0 \cdots y_n & \\ & \to (x < fx \wedge ((y_i \equiv \bar{0} \wedge Rfx\overline{L+1}y_0 \cdots y_n) \\ & \qquad \vee (\neg y_i \equiv \bar{0} \wedge \exists u(fu \equiv y_i \\ & \qquad\qquad \wedge Rfx\overline{L+1}y_0 \cdots y_{i-1}uy_{i+1} \cdots y_n)))))\big).\end{aligned}$$
- $\alpha = L$ **IF** $R_i = \square$ **THEN** L' **ELSE** L_0. Then let
$$\begin{aligned}\psi_\alpha := \forall x \forall y_0 \cdots \forall y_n \big(Rx\bar{L}y_0 \cdots y_n & \\ & \to (x < fx \wedge ((y_i \equiv \bar{0} \wedge Rfx\overline{L'}y_0 \cdots y_n) \\ & \qquad \vee (\neg y_i \equiv \bar{0} \wedge Rfx\overline{L_0}y_0 \cdots y_n)))\big).\end{aligned}$$
- $\alpha = L$ **PRINT**. Then let
$$\psi_\alpha := \forall x \forall y_0 \cdots \forall y_n \big(Rx\bar{L}y_0 \cdots y_n \to (x < fx \wedge Rfx\overline{L+1}y_0 \cdots y_n)\big).$$

The verification of (P1) and (P2) are left as an exercise.

Finally let
$$\varphi_{\mathbb{P}} := \psi_{\mathbb{P}} \to \exists x \exists y_0 \cdots \exists y_n Rx\bar{k}y_0 \cdots y_n.$$
Now we verify that $\mathbb{P} : \square \to \text{halt}$ if and only if $\models \varphi_{\mathbb{P}}$. First, assume $\models \varphi_{\mathbb{P}}$, in particular
$$\mathfrak{A}_{\mathbb{P}} \models \varphi_{\mathbb{P}}.$$
By (P1) we conclude
$$\mathfrak{A}_{\mathbb{P}} \models \exists x \exists y_0 \cdots \exists y_n Rx\bar{k}y_0 \cdots y_n.$$
Then there are some $s, m_0, \ldots, m_n \in A_{\mathbb{P}} \subseteq \mathbb{N}$ such that $(k, m_0, \ldots, m_n)$ is the configuration of $\mathbb{P}$ after s steps. Therefore, $\mathbb{P}$ reaches the last halt instruction after s steps, hence $\mathbb{P} : \square \to \text{halt}$.

Conversely, assume $\mathbb{P} : \square \to \text{halt}$. Let $\mathfrak{A}$ be an S-structure. We need to show that $\mathfrak{A} \models \varphi_{\mathbb{P}}$. Clearly, if $\mathfrak{A} \not\models \psi_{\mathbb{P}}$, then we are already done. Thus, assume $\mathfrak{A} \models \psi_{\mathbb{P}}$. Recall that $s_{\mathbb{P}} \in \mathbb{N}$ is the number of steps which $\mathbb{P}$ carries out until it reaches the last halt instruction α_k. Hence, for some $m_0, \ldots, m_n \leqslant s_{\mathbb{P}}$ the tuple
$$(k, m_0, \ldots, m_n)$$
is the configuration of $\mathbb{P}$ after $s_{\mathbb{P}}$ steps. Now (P2) implies that
$$\mathfrak{A} \models R\overline{s_{\mathbb{P}}}\bar{k}\bar{m}_0 \cdots \bar{m}_n.$$
Therefore
$$\mathfrak{A} \models \varphi_{\mathbb{P}}.$$
This finishes the proof. □

2. Exercises

Exercise 2.1. Prove (P1) and (P2) in the proof of Theorem 1.2. ⊣

Exercise 2.2. Assume $\mathbb{P} : \square \to \text{halt}$. Construct an *infinite* S-structure with $\mathfrak{A} \models \psi_{\mathbb{P}}$.

Exercise 2.3. Show that
$$\{\varphi \in L_0^{S_\infty} \mid \varphi \text{ is satisfiable}\}$$
is not R-enumerable. ⊣