

# Mathematical Logic (XIV)

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## 1. Gödel's Incompleteness Theorems

Let  $\Phi \subseteq L_0^{\text{Sar}}$ .

**Definition 1.1.** Let  $r \geq 1$ .

- (i) A relation  $\mathcal{R} \subseteq \mathbb{N}^r$  is *representable in  $\Phi$*  if there is an  $L^{\text{Sar}}$ -formula  $\varphi(v_0, \dots, v_{r-1})$  such that for all  $n_0, \dots, n_{r-1} \in \mathbb{N}$

$$\begin{aligned} (n_0, \dots, n_{r-1}) \in \mathcal{R} &\implies \Phi \vdash \varphi(\bar{n}_0, \dots, \bar{n}_{r-1}), \\ (n_0, \dots, n_{r-1}) \notin \mathcal{R} &\implies \Phi \vdash \neg\varphi(\bar{n}_0, \dots, \bar{n}_{r-1}). \end{aligned}$$

- (ii) A function  $F : \mathbb{N}^r \rightarrow \mathbb{N}$  is *representable in  $\Phi$*  if there is an  $L^{\text{Sar}}$ -formula  $\varphi(v_0, \dots, v_{r-1}, v_r)$  such that for all  $n_0, \dots, n_{r-1}, n_r \in \mathbb{N}$

$$\begin{aligned} f(n_0, \dots, n_{r-1}) = n_r &\implies \Phi \vdash \varphi(\bar{n}_0, \dots, \bar{n}_{r-1}, \bar{n}_r), \\ f(n_0, \dots, n_{r-1}) \neq n_r &\implies \Phi \vdash \neg\varphi(\bar{n}_0, \dots, \bar{n}_{r-1}, \bar{n}_r). \end{aligned}$$

Moreover,

$$\Phi \vdash \exists^{=1} v_r \varphi(\bar{n}_0, \dots, \bar{n}_{r-1}, v_r). \quad \dashv$$

**Lemma 1.2.** (i) If  $\Phi$  is inconsistent, then every relation over  $\mathbb{N}$  and every function over  $\mathbb{N}$  is representable in  $\Phi$ .

- (ii) Let  $\Phi \subseteq \Phi' \subseteq L_0^{\text{Sar}}$ . Then every relation representable in  $\Phi$  is also representable in  $\Phi'$ . Similarly, every function representable in  $\Phi$  is representable in  $\Phi'$  as well.

- (iii) Let  $\Phi$  be consistent. If  $\Phi$  is R-decidable, then every relation representable in  $\Phi$  is R-decidable, and every function representable in  $\Phi$  is R-computable.  $\dashv$

**Definition 1.3.**  $\Phi$  *allows representations* if all R-decidable relations and all R-computable functions over  $\mathbb{N}$  are representable in  $\Phi$ .  $\dashv$

**Theorem 1.4.**  $\text{Th}(\mathfrak{N})$  *allows representations*.  $\dashv$

**Theorem 1.5.**  $\Phi_{\text{PA}}$  *allows representations*.  $\dashv$

Recall that we have exhibited the so-called Gödel numbering of register programs. For later purposes, we do the same for  $L^{\text{Sar}}$ -formulas. Let

$$\varphi_0, \varphi_1, \dots, \quad (1)$$

be an *effective* enumeration of all  $L^{\text{Sar}}$ -formulas without repetition. That is, there is a program that prints out the sequence (1). Then for every  $\varphi \in L^{\text{Sar}}$  we let

$$[\varphi] := n \quad \text{where } \varphi = \varphi_n.$$

Observe that both

$$n \mapsto \varphi_n \quad \text{and} \quad \varphi \mapsto [\varphi]$$

are R-computable.

**Theorem 1.6** (Fixed Point Theorem). *Assume that  $\Phi$  allows representations. Then for every  $\psi \in L_1^{S_{ar}}$ , there is an  $S_{ar}$ -sentence  $\varphi$  such that*

$$\Phi \vdash \varphi \leftrightarrow \psi(\overline{[\varphi]}). \quad (2)$$

*Proof:* We define a function  $F : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  as follows. For every  $n, m \in \mathbb{N}$

$$F(n, m) := \begin{cases} [\varphi_n(\bar{m})] & \text{if } \text{free}(\varphi_n) = \{v_0\}, \\ & \text{i.e., } \varphi_n \in L_1^{S_{ar}} \setminus L_0^{S_{ar}}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that  $F$  is R-computable, and for every  $\varphi \in L_1^{S_{ar}} \setminus L_0^{S_{ar}}$  we have

$$F([\varphi], m) = [\varphi(\bar{m})]. \quad (3)$$

Since  $\Phi$  allows representations, there is an  $S_{ar}$ -formula  $\varphi_F(x, y, z)$  such that for all  $n, m, \ell \in \mathbb{N}$

$$F(n, m) = \ell \implies \Phi \vdash \varphi_F(\bar{n}, \bar{m}, \bar{\ell}), \quad (4)$$

$$F(n, m) \neq \ell \implies \Phi \vdash \neg \varphi_F(\bar{n}, \bar{m}, \bar{\ell}). \quad (5)$$

Moreover,

$$\Phi \vdash \exists^{=1} z \varphi_F(\bar{n}, \bar{m}, z). \quad (6)$$

Let

$$\chi(v_0) := \forall x (\varphi_F(v_0, v_0, x) \rightarrow \psi(x)).$$

In particular,  $\text{free}(\chi) = \{v_0\}$ . Finally we define the desired

$$\varphi := \chi(\bar{n}) \quad \text{with } n = [\chi].$$

We show that (2) holds. First, by (3)

$$F(n, n) = F([\chi], n) = [\chi(\bar{n})] = [\varphi].$$

Then (4) implies

$$\Phi \vdash \varphi_F(\bar{n}, \bar{n}, \overline{[\varphi]}) \quad (7)$$

Recall

$$\varphi = \chi(\bar{n}) = \forall x (\varphi_F(\bar{n}, \bar{n}, x) \rightarrow \psi(x)).$$

Combined with (7) we obtain

$$\Phi \cup \{\varphi\} \vdash \psi(\overline{[\varphi]}).$$

Equivalently

$$\Phi \vdash \varphi \rightarrow \psi(\overline{[\varphi]}).$$

For the other direction in (2), observe that (6) and (7) guarantee that

$$\Phi \vdash \forall z (\varphi(\bar{n}, \bar{n}, z) \rightarrow z \equiv \overline{[\varphi]}).$$

Thus

$$\Phi \cup \{\psi(\overline{[\varphi]})\} \vdash \forall x (\varphi_F(\bar{n}, \bar{n}, x) \rightarrow \psi(x)),$$

i.e.,  $\Phi \cup \{\psi(\overline{[\varphi]})\} \vdash \varphi$ . It follows that

$$\Phi \vdash \psi(\overline{[\varphi]}) \rightarrow \varphi. \quad \square$$

**Definition 1.7.** Let  $\Phi \subseteq L^{\text{Sar}}$ . Then

$$\Phi^{\vdash} := \{\varphi \in L^{\text{Sar}} \mid \Phi \vdash \varphi\}.$$

We say that  $\Phi^{\vdash}$  is *representable in  $\Phi$*  if

$$\{[\varphi] \in \mathbb{N} \mid \varphi \in \Phi^{\vdash}\} = \{[\varphi] \mid \varphi \in L^{\text{Sar}} \text{ and } \Phi \vdash \varphi\}.$$

is representable in  $\Phi$ . □

**Lemma 1.8.** Let  $\Phi \subseteq L^{\text{Sar}}$  be consistent and allow representations. Then  $\Phi^{\vdash}$  is not representable in  $\Phi$ .

*Proof:* Assume that  $\Phi^{\vdash}$  is representable in  $\Phi$ . In particular, there is a  $\chi(v_0) \in L_1^{\text{Sar}}$  such that for all  $\varphi \in L_0^{\text{Sar}}$

$$\begin{aligned} \Phi \vdash \varphi &\implies \Phi \vdash \chi(\overline{[\varphi]}), \\ \Phi \not\vdash \varphi &\implies \Phi \vdash \neg\chi(\overline{[\varphi]}). \end{aligned}$$

Since  $\Phi$  is consistent, we conclude

$$\Phi \not\vdash \varphi \iff \Phi \vdash \neg\chi(\overline{[\varphi]}). \quad (8)$$

We apply the Fixed Point Theorem 1.6 to  $\neg\chi$  to obtain a sentence  $\varphi$  such that

$$\Phi \vdash \varphi \leftrightarrow \neg\chi(\overline{[\varphi]}). \quad (9)$$

Then

$$\begin{aligned} \Phi \vdash \varphi &\iff \Phi \vdash \neg\chi(\overline{[\varphi]}) && \text{(by (9))} \\ &\iff \Phi \not\vdash \varphi, && \text{(by (8))} \end{aligned}$$

which is a contradiction. □

**Theorem 1.9** (Tarski's Undefinability of the Arithmetic Truth).

- (i) Let  $\Phi \subseteq L^{\text{Sar}}$  be consistent and allow representations. Then  $\Phi^{\models}$  is not representable in  $\Phi$ .
- (ii)  $\text{Th}(\mathfrak{N})$  is not representable in  $\text{Th}(\mathfrak{N})$ .

*Proof:* By the Completeness Theorem

$$\Phi^{\models} = \Phi^{\vdash}.$$

So (i) is a direct consequence of Lemma 1.8.

(ii) is a special case of (i). □

**Theorem 1.10** (Gödel's First Incompleteness Theorem). Let  $\Phi \subseteq L^{\text{Sar}}$  be consistent and allow representations. Moreover,  $\Phi$  is R-decidable. Then there is an  $L^{\text{Sar}}$ -sentence  $\varphi$  such that neither  $\Phi \vdash \varphi$  nor  $\Phi \vdash \neg\varphi$ .

*Proof:* Assume for every  $L^{\text{Sar}}$ -sentence  $\varphi$  either  $\Phi \vdash \varphi$  or  $\Phi \vdash \neg\varphi$ . Thus  $\Phi$  is complete. By the R-decidability of  $\Phi$ , we can then conclude that  $\Phi^{\vdash}$  is R-decidable too.

Since  $\Phi$  allows representations,  $\Phi^{\vdash}$  is representable in  $\Phi$ . Together with the consistency of  $\Phi$ , we obtain a contradiction to Lemma 1.8. □

In the following we fix an R-decidable  $\Phi \subseteq L_0^{\text{Sar}}$  which allows representations.

We choose an effective enumeration of all derivations in the sequent calculus associated with  $S_{\text{ar}}$  and define a relation  $\mathcal{H} \subseteq \mathbb{N}^2$  by

$$(n, m) \in \mathcal{H} \iff \text{the } m\text{-th derivation in the above enumeration ends with a sequent } \psi_0, \dots, \psi_{k-1}, \varphi \text{ with } \psi_0, \dots, \psi_{k-1} \in \Phi \text{ and } n = [\varphi],$$

Clearly,  $\mathcal{H}$  is R-decidable by the R-decidability of  $\Phi$ . Moreover, for every  $\varphi \in L^{S_{\text{ar}}}$

$$\Phi \vdash \varphi \iff \text{there is an } m \in \mathbb{N} \text{ with } ([\varphi], m) \in \mathcal{H}.$$

Since  $\Phi$  allows representation, there is a  $\varphi_{\mathcal{H}}(v_0, v_1) \in L_2^{S_{\text{ar}}}$  such that for every  $n, m \in \mathbb{N}$

$$(n, m) \in \mathcal{H} \implies \Phi \vdash \varphi_{\mathcal{H}}(\bar{n}, \bar{m}), \quad (10)$$

$$(n, m) \notin \mathcal{H} \implies \Phi \vdash \neg \varphi_{\mathcal{H}}(\bar{n}, \bar{m}). \quad (11)$$

We set

$$\text{DER}_{\Phi}(x) := \exists y \varphi_{\mathcal{H}}(x, y),$$

which intuitively says that  $x$  is provable in  $\Phi$ .

Applying Lemma 1.6 to  $\psi(x) := \neg \text{DER}_{\Phi}(x)$ , we obtain an  $L_0^{S_{\text{ar}}}$ -sentence  $\varphi$  such that

$$\Phi \vdash \varphi \leftrightarrow \neg \text{DER}_{\Phi}([\overline{\varphi}]). \quad (12)$$

**Lemma 1.11.** *If  $\Phi$  is consistent, then  $\Phi \not\vdash \varphi$ .*

*Proof:* Assume that  $\Phi \vdash \varphi$ , which is given by the  $m$ -th derivation for some  $m \in \mathbb{N}$ . In other words,

$$([\varphi], m) \in \mathcal{H}.$$

Then, (10) implies

$$\Phi \vdash \varphi_{\mathcal{H}}([\overline{\varphi}], \bar{m}).$$

It follows that

$$\Phi \vdash \text{DER}_{\Phi}([\overline{\varphi}]).$$

By (12)

$$\Phi \vdash \neg \varphi.$$

Thus  $\Phi$  is inconsistent. □

Observe that  $\Phi \vdash 0 \equiv 0$ , therefore

$$\Phi \text{ is consistent} \iff \Phi \not\vdash \neg 0 \equiv 0.$$

Hence,

$$\text{CONS}_{\Phi} := \neg \text{DER}_{\Phi}([\neg 0 \equiv 0])$$

expresses that  $\Phi$  is consistent.

**Lemma 1.12.** *Assume  $\Phi_{\text{PA}} \subseteq \Phi$ . Then*

$$\Phi \vdash \text{CONS}_{\Phi} \rightarrow \neg \text{DER}_{\Phi}([\varphi]),$$

where  $\varphi$  is the sentence in (12).

*Proof:* A tedious analysis shows that the proof of Lemma 12 can be carried out on the basis of  $\Phi_{\text{PA}}$ . □

**Theorem 1.13** (Gödel's Second Incompleteness Theorem). *Assume  $\Phi$  is consistent and R-decidable with  $\Phi_{\text{PA}} \subseteq \Phi$ . Then*

$$\Phi \not\vdash \text{CONS}_{\Phi}.$$

*Proof:* Assume  $\Phi \vdash \text{CONS}_{\Phi}$ . Then Lemma 1.12 implies

$$\Phi \vdash \neg \text{DER}_{\Phi}([\varphi]).$$

By (12) we have

$$\Phi \vdash \varphi,$$

which contradicts to Lemma 12. □