

Mathematical Logic (II)

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1. The Semantics of First-order Logic

1.1. Structures and interpretations. We fix a symbol set S .

Definition 1.1. An S -structure is a pair $\mathfrak{A} = (A, \alpha)$ which satisfies the following conditions.

- (1) $A \neq \emptyset$ is the *universe* of \mathfrak{A} .
- (2) α is a function defined on S such that:
 - (a) Let $R \in S$ be an n -ary relation symbol. Then $\alpha(R) \subseteq A^n$.
 - (b) Let $f \in S$ be an n -ary function symbol. Then $\alpha(f) : A^n \rightarrow A$.
 - (c) $\alpha(c) \in A$ for every constant $c \in S$.

For better readability, we write $R^{\mathfrak{A}}$, $f^{\mathfrak{A}}$, and $c^{\mathfrak{A}}$, or even R^A , f^A , and c^A , instead of $\alpha(R)$, $\alpha(f)$, and $\alpha(c)$. Thus for $S = \{R, f, c\}$ we might write an S -structure as

$$\mathfrak{A} = (A, R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}) = (A, R^A, f^A, c^A). \quad \dashv$$

Examples 1.2. (1) For $S_{Ar} := \{+, \cdot, 0, 1\}$ the S_{Ar} structure

$$\mathfrak{N} = (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}})$$

is the standard model of natural numbers with addition, multiplication, and constants 0 and 1.

(2) For $S_{Ar}^< := \{+, \cdot, 0, 1, <\}$ we have an $S_{Ar}^<$ structure

$$\mathfrak{N}^< = (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, <^{\mathbb{N}}),$$

i.e., the standard model of \mathbb{N} with the natural ordering $<$. \dashv

Definition 1.3. An *assignment* in an S -structure \mathfrak{A} is a mapping

$$\beta : \{v_i \mid i \in \mathbb{N}\} \rightarrow A. \quad \dashv$$

Definition 1.4. An S -interpretation \mathfrak{I} is a pair (\mathfrak{A}, β) where \mathfrak{A} is an S -structure and β is an assignment in \mathfrak{A} . \dashv

Definition 1.5. Let β be an assignment in \mathfrak{A} , $a \in A$, and x a variable. Then $\beta \frac{a}{x}$ is the assignment defined by

$$\beta \frac{a}{x}(y) := \begin{cases} a, & \text{if } y = x, \\ \beta(y), & \text{otherwise.} \end{cases}$$

Then, for the S -interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ we use $\mathfrak{I} \frac{a}{x}$ to denote the S -interpretation $(\mathfrak{A}, \beta \frac{a}{x})$. \dashv

Definition 1.6. Let \mathfrak{A} and \mathfrak{B} be two S -structures. Their *direct product* $\mathfrak{A} \times \mathfrak{B}$ is the S -structure defined as follows.

– The universe of $\mathfrak{A} \times \mathfrak{B}$ is $A \times B$.

– For every n-ary relation symbol $R \in S$

$$R^{\mathfrak{A} \times \mathfrak{B}} := \{((a_1, b_1), \dots, (a_n, b_n)) \mid (a_1, \dots, a_n) \in R^{\mathfrak{A}} \text{ and } (b_1, \dots, b_n) \in R^{\mathfrak{B}}\}.$$

– For every n-ary function symbol $f \in S$

$$f^{\mathfrak{A} \times \mathfrak{B}}((a_1, b_1), \dots, (a_n, b_n)) := (f^{\mathfrak{A}}(a_1, \dots, a_n), f^{\mathfrak{B}}(b_1, \dots, b_n)).$$

– For every constant $c \in S$

$$c^{\mathfrak{A} \times \mathfrak{B}} := (c^{\mathfrak{A}}, c^{\mathfrak{B}}). \quad \dashv$$

1.2. The satisfaction relation $\mathfrak{I} \models \varphi$. We fix an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$.

Definition 1.7. For every S-term t we define its *interpretation* $\mathfrak{I}(t)$ by induction on the construction of t .

(a) $\mathfrak{I}(x) = \beta(x)$ for a variable x .

(b) $\mathfrak{I}(c) = c^{\mathfrak{A}}$ for a constant $c \in S$.

(c) Let $f \in S$ be an n-ary function symbol and t_1, \dots, t_n S-terms. Then

$$\mathfrak{I}(ft_1 \cdots t_n) = f^{\mathfrak{A}}(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)). \quad \dashv$$

Example 1.8. Let $S := S_{Gr} = \{\circ, e\}$ and $\mathfrak{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$, $\beta(v_0) = 2$, and $\beta(v_2) = 6$. Then

$$\begin{aligned} \mathfrak{I}(v_0 \circ (e \circ v_2)) &= \mathfrak{I}(v_0) + \mathfrak{I}(e \circ v_2) \\ &= 2 + (\mathfrak{I}(e) + \mathfrak{I}(v_2)) = 2 + (0 + 6) = 2 + 6 = 8. \end{aligned} \quad \dashv$$

Definition 1.9. Let φ be an S-formula. We define $\mathfrak{I} \models \varphi$ by induction on the construction of φ .

(a) $\mathfrak{I} \models t_1 \equiv t_2$ if $\mathfrak{I}(t_1) = \mathfrak{I}(t_2)$.

(b) $\mathfrak{I} \models R t_1 \cdots t_n$ if $(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)) \in R^{\mathfrak{A}}$.

(c) $\mathfrak{I} \models \neg \varphi$ if $\mathfrak{I} \not\models \varphi$ (i.e., it is *not* the case that $\mathfrak{I} \models \varphi$).

(d) $\mathfrak{I} \models (\varphi \wedge \psi)$ if $\mathfrak{I} \models \varphi$ and $\mathfrak{I} \models \psi$.

(e) $\mathfrak{I} \models (\varphi \vee \psi)$ if $\mathfrak{I} \models \varphi$ or $\mathfrak{I} \models \psi$.

(f) $\mathfrak{I} \models (\varphi \rightarrow \psi)$ if $\mathfrak{I} \models \varphi$ implies $\mathfrak{I} \models \psi$.

(g) $\mathfrak{I} \models (\varphi \leftrightarrow \psi)$ if $(\mathfrak{I} \models \varphi \text{ if and only if } \mathfrak{I} \models \psi)$.

(h) $\mathfrak{I} \models \forall x \varphi$ if for all $a \in A$ we have $\mathfrak{I} \frac{a}{x} \models \varphi$.

(i) $\mathfrak{I} \models \exists x \varphi$ if for some $a \in A$ we have $\mathfrak{I} \frac{a}{x} \models \varphi$.

If $\mathfrak{I} \models \varphi$, then \mathfrak{I} is a *model* of φ , of \mathfrak{I} *satisfies* φ .

Let Φ be a set of S-formulas. Then $\mathfrak{I} \models \Phi$ if $\mathfrak{I} \models \varphi$ for all $\varphi \in \Phi$. Similarly as above, we say that \mathfrak{I} is a model of Φ , or \mathfrak{I} satisfies Φ . \dashv

Example 1.10. Let $S := S_{Gr}$ and $\mathfrak{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$ and $\beta(x) = 9$ for all variables x . Then

$$\begin{aligned} \mathfrak{I} \models \forall v_0 v_0 \circ e \equiv v_0 &\iff \text{for all } r \in \mathbb{R} \text{ we have } \mathfrak{I} \frac{r}{v_0} \models v_0 \circ e \equiv v_0, \\ &\iff \text{for all } r \in \mathbb{R} \text{ we have } r + 0 = r. \end{aligned} \quad \dashv$$

Definition 1.11. Let Φ be a set of S-formulas and φ an S-formula. Then φ is a *consequence* of Φ , written $\Phi \models \varphi$, if for any interpretation \mathcal{I} it holds that $\mathcal{I} \models \Phi$ implies $\mathcal{I} \models \varphi$.

For simplicity, in case $\Phi = \{\psi\}$ we write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$. ⊢

Example 1.12. Let

$$\Phi_{Gr} := \{ \forall v_0 \forall v_1 \forall v_2 (v_0 \circ v_1) \circ v_2 \equiv v_0 \circ (v_1 \circ v_2), \\ \forall v_0 v_0 \circ e \equiv v_0, \forall v_0 \exists v_1 v_0 \circ v_1 \equiv e \}.$$

Then it can be shown that

$$\Phi_{Gr} \models \forall v_0 e \circ v_0 \equiv v_0.$$

and

$$\Phi_{Gr} \models \forall v_0 \exists v_1 v_1 \circ v_0 \equiv e. \quad \dashv$$

Definition 1.13. An S-formula φ is *valid*, written $\models \varphi$, if $\emptyset \models \varphi$. Or equivalently, $\mathcal{I} \models \varphi$ for any \mathcal{I} . ⊢

Definition 1.14. An S-formula φ is *satisfiable*, if there exists an S-interpretation \mathcal{I} with $\mathcal{I} \models \varphi$. A set Φ of S-formulas is satisfiable if there exists an S-interpretation \mathcal{I} such that $\mathcal{I} \models \varphi$ for every $\varphi \in \Phi$. ⊢

The next lemma is essentially the method of *proof by contradiction*.

Lemma 1.15. Let Φ be a set of S-formulas and φ an S-formula. Then $\Phi \models \varphi$ if and only if $\Phi \cup \{\neg\varphi\}$ is not satisfiable. ⊢

Proof:

$$\begin{aligned} \Phi \models \varphi &\iff \text{Every model of } \Phi \text{ is a model of } \varphi, \\ &\iff \text{there is no model } \mathcal{I} \text{ with } \mathcal{I} \models \Phi \text{ and } \mathcal{I} \not\models \varphi, \\ &\iff \text{there is no model } \mathcal{I} \text{ with } \mathcal{I} \models \Phi \cup \{\neg\varphi\}, \\ &\iff \Phi \cup \{\neg\varphi\} \text{ is not satisfiable.} \quad \square \end{aligned}$$

Definition 1.16. Two S-formulas φ and ψ are *logic equivalent* if $\varphi \models \psi$ and $\psi \models \varphi$. ⊢

Example 1.17. Let φ be an S-formula. We define a logic equivalent φ^* which does not contain the logic symbols $\wedge, \rightarrow, \leftrightarrow, \forall$.

$$\begin{aligned} \varphi^* &:= \varphi \quad \text{if } \varphi \text{ is atomic,} \\ (\neg\varphi)^* &:= \neg\varphi^*, \\ (\varphi \wedge \psi)^* &:= \neg(\neg\varphi^* \vee \neg\psi^*), \\ (\varphi \vee \psi)^* &:= (\varphi^* \vee \psi^*), \\ (\varphi \rightarrow \psi)^* &:= (\neg\varphi^* \vee \psi^*), \\ (\varphi \leftrightarrow \psi)^* &:= \neg(\varphi^* \vee \psi^*) \vee \neg(\neg\varphi^* \vee \neg\psi^*), \\ (\forall x\varphi)^* &:= \neg\exists x\neg\varphi^*, \\ (\exists x\varphi)^* &:= \exists x\varphi^*. \end{aligned}$$

Thus, it suffices to consider \neg, \vee, \exists as the only logic symbols in any given φ . ⊢

2. Exercises

Exercise 2.1. Prove that:

- (a) If \mathfrak{A} and \mathfrak{B} are both groups, then so is $\mathfrak{A} \times \mathfrak{B}$.
- (b) If \mathfrak{A} and \mathfrak{B} are both equivalence relations, then so is $\mathfrak{A} \times \mathfrak{B}$.
- (c) For two fields \mathfrak{A} and \mathfrak{B} , their direct product $\mathfrak{A} \times \mathfrak{B}$ is not necessarily a field. +

Exercise 2.2. Prove Example 1.12.

Exercise 2.3. An S-formula is *positive* if it contains no logic symbols \neg , \rightarrow , and \leftrightarrow . Prove that every positive formula is satisfiable.