

Mathematical Logic (III)

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1. The Semantics of First-order Logic

Lemma 1.1 (The Coincidence Lemma). *For $i \in \{1, 2\}$ let $\mathcal{I}_i = (\mathfrak{A}_i, \beta_i)$ be an S_i -interpretation such that $A_1 = A_2$ and every symbol in $S := S_1 \cap S_2$ has the same interpretation in \mathfrak{A}_1 and \mathfrak{A}_2 .*

- (a) *Let t be an S -term (thus also an S_1 -term and an S_2 -term). Assume further that $\beta_1(x) = \beta_2(x)$ for every variable $x \in \text{var}(t)$. Then $\mathcal{I}_1(t) = \mathcal{I}_2(t)$.*
- (b) *Let φ be an S -formula where $\beta_1(x) = \beta_2(x)$ for every $x \in \text{free}(\varphi)$. Then*

$$\mathcal{I}_1 \models \varphi \iff \mathcal{I}_2 \models \varphi.$$

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Proof: (a) We prove by induction on t .

- $t = x$. Then $\mathcal{I}_1(x) = \beta_1(x) = \beta_2(x) = \mathcal{I}_2(x)$.
- $t = c$. We deduce $\mathcal{I}_1(c) = c^{\mathfrak{A}_1} = c^{\mathfrak{A}_2} = \mathcal{I}_2(c)$.
- $t = ft_1 \cdots t_n$. It holds that

$$\begin{aligned} \mathcal{I}_1(ft_1 \cdots t_n) &= f^{\mathfrak{A}_1}(\mathcal{I}_1(t_1), \dots, \mathcal{I}_2(t_n)) \\ &= f^{\mathfrak{A}_2}(\mathcal{I}_1(t_1), \dots, \mathcal{I}_1(t_n)) \\ &= f^{\mathfrak{A}_2}(\mathcal{I}_2(t_1), \dots, \mathcal{I}_2(t_n)) \\ &= \mathcal{I}_2(ft_1 \cdots t_n). \end{aligned}$$

(b) The induction proof is on the structure of φ .

- $\varphi = t_1 \equiv t_2$. We have

$$\begin{aligned} \mathcal{I}_1 \models t_1 \equiv t_2 &\iff \mathcal{I}_1(t_1) = \mathcal{I}_1(t_2) \\ &\iff \mathcal{I}_2(t_1) = \mathcal{I}_2(t_2) && \text{(by (a))} \\ &\iff \mathcal{I}_2 \models t_1 \equiv t_2. \end{aligned}$$

- $\varphi = R t_1 \cdots t_n$. Then

$$\begin{aligned} \mathcal{I}_1 \models R t_1 \cdots t_n &\iff (\mathcal{I}_1(t_1), \dots, \mathcal{I}_1(t_n)) \in R^{\mathfrak{A}_1} \\ &\iff (\mathcal{I}_1(t_1), \dots, \mathcal{I}_1(t_n)) \in R^{\mathfrak{A}_2} \\ &\iff (\mathcal{I}_2(t_1), \dots, \mathcal{I}_2(t_n)) \in R^{\mathfrak{A}_2} \\ &\iff \mathcal{I}_2 \models R t_1 \cdots t_n. \end{aligned}$$

- $\varphi = \neg \psi$. We conclude

$$\mathcal{I}_1 \models \neg \psi \iff \mathcal{I}_1 \not\models \psi \iff \mathcal{I}_2 \not\models \psi \iff \mathcal{I}_2 \models \neg \psi.$$

- $\varphi = (\psi \vee \chi)$.

$$\begin{aligned} \mathcal{J}_1 \models (\psi \vee \chi) &\iff \mathcal{J}_1 \models \psi \text{ or } \mathcal{J}_1 \models \chi \\ &\iff \mathcal{J}_2 \models \psi \text{ or } \mathcal{J}_2 \models \chi \\ &\iff \mathcal{J}_2 \models (\psi \vee \chi). \end{aligned}$$

- $\varphi = \exists x\psi$.

$$\begin{aligned} \mathcal{J}_1 \models \exists x\psi &\iff \text{for some } a \in A_1 \text{ we have } \mathcal{J}_1 \frac{a}{x} \models \psi \\ &\iff \text{for some } a \in A_1 \text{ we have } \mathcal{J}_2 \frac{a}{x} \models \psi \\ &\quad \left(\text{by induction hypothesis on } \mathcal{J}_1 \frac{a}{x}, \mathcal{J}_2 \frac{a}{x}, \text{ and } \psi \right) \\ &\iff \mathcal{J}_2 \models \exists x\psi. \end{aligned}$$

□

Remark 1.2. Let $\varphi \in L_n^S$, i.e., φ is an S-formula with $\text{free}(\varphi) \subseteq \{v_0, \dots, v_{n-1}\}$. By the coincidence lemma whether $\mathcal{J} = (\mathfrak{A}, \beta) \models \varphi$ is completely determined by \mathfrak{A} and $\beta(v_0), \dots, \beta(v_{n-1})$. So in case $\mathcal{J} \models \varphi$ we can write

$$\mathfrak{A} \models \varphi[a_0, \dots, a_{n-1}]$$

where $a_i := \beta(v_i)$ for $0 \leq i < n$. In particular, if φ is an S-sentence, i.e., $\varphi \in L_0^S$, then $\mathfrak{A} \models \varphi$ is well-defined.

Similarly, we write

$$t^{\mathfrak{A}}[a_0, \dots, a_{n-1}]$$

instead of $\mathcal{J}(t)$.

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Definition 1.3. Let \mathfrak{A} and \mathfrak{B} be two S-structures.

(a) A mapping $\pi : A \rightarrow B$ is an *isomorphism from \mathfrak{A} to \mathfrak{B}* (in short $\pi : \mathfrak{A} \cong \mathfrak{B}$) if the following conditions are satisfied.

(i) π is a bijection.

(ii) For any n-ary relation symbol $R \in S$ and $a_0, \dots, a_{n-1} \in A$

$$(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \iff (\pi(a_0), \dots, \pi(a_{n-1})) \in R^{\mathfrak{B}}.$$

(iii) For any n-ary function symbol $f \in S$ and $a_0, \dots, a_{n-1} \in A$

$$\pi(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{B}}(\pi(a_0), \dots, \pi(a_{n-1})).$$

(iv) For any constant $c \in S$

$$\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

(b) \mathfrak{A} and \mathfrak{B} are isomorphic, written $\mathfrak{A} \cong \mathfrak{B}$, if there is an isomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$.

⊢

Observe that the above definition is not symmetric. However we can easily show:

Lemma 1.4. \cong is an equivalence relation. That is, for all S-structures $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$

(1) $\mathfrak{A} \cong \mathfrak{A}$;

(2) $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{B} \cong \mathfrak{A}$;

(3) if $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{B} \cong \mathfrak{C}$, then $\mathfrak{A} \cong \mathfrak{C}$.

⊢

Lemma 1.5 (The Isomorphism Lemma). *Let \mathfrak{A} and \mathfrak{B} be two isomorphic S-structures. Then for every S-sentence φ*

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi.$$

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Proof: Let β be an assignment in \mathfrak{A} . By the coincidence lemma, it suffices to show that there is an assignment β' in \mathfrak{B} such that

$$(\mathfrak{A}, \beta) \models \varphi \iff (\mathfrak{B}, \beta') \models \varphi, \quad (1)$$

where φ is an S-sentence.

Let $\pi : \mathfrak{A} \cong \mathfrak{B}$ and we define an assignment β^π in \mathfrak{B} by

$$\beta^\pi(x) := \pi(\beta(x))$$

for any variable x . Then we prove for any S-formula φ

$$(\mathfrak{A}, \beta) \models \varphi \iff (\mathfrak{B}, \beta^\pi) \models \varphi, \quad (2)$$

which certainly generalizes (1). To simplify notation, let $\mathfrak{J} := (\mathfrak{A}, \beta)$ and $\mathfrak{J}^\pi := (\mathfrak{B}, \beta^\pi)$. First, it is routine to verify that for every S-term t

$$\pi(\mathfrak{J}(t)) = \mathfrak{J}^\pi(t). \quad (3)$$

Then we prove (2) by induction on the construction of S-formula φ .

– $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned} \mathfrak{J} \models t_1 \equiv t_2 &\iff \mathfrak{J}(t_1) = \mathfrak{J}(t_2) \\ &\iff \pi(\mathfrak{J}(t_1)) = \pi(\mathfrak{J}(t_2)) && \text{(since } \pi \text{ is an injection)} \\ &\iff \mathfrak{J}^\pi(t_1) = \mathfrak{J}^\pi(t_2) && \text{(by (3))} \\ &\iff \mathfrak{J}^\pi \models t_1 \equiv t_2. \end{aligned}$$

– $\varphi = R t_1 \cdots t_n$.

$$\begin{aligned} \mathfrak{J} \models R t_1 \cdots t_n &\iff (\mathfrak{J}(t_1), \dots, \mathfrak{J}(t_n)) \in R^{\mathfrak{A}} \\ &\iff (\pi(\mathfrak{J}(t_1)), \dots, \pi(\mathfrak{J}(t_n))) \in R^{\mathfrak{B}} \\ &\iff (\mathfrak{J}^\pi(t_1), \dots, \mathfrak{J}^\pi(t_n)) \in R^{\mathfrak{B}} && \text{(by (3))} \\ &\iff \mathfrak{J}^\pi \models R t_1 \cdots t_n. \end{aligned}$$

– $\varphi = \neg\psi$. It follows that $\mathfrak{J} \models \neg\psi \iff \mathfrak{J} \not\models \psi \iff \mathfrak{J}^\pi \not\models \psi \iff \mathfrak{J}^\pi \models \neg\psi$.

– $\varphi = \psi \vee \chi$. The inductive argument is similar to the above $\neg\psi$.

– $\varphi = \exists x\psi$. This is again the most complicated case.

$$\begin{aligned} \mathfrak{J} \models \exists x\psi &\iff \text{there exists an } a \in A \text{ such that } \mathfrak{J} \frac{a}{x} = \left(\mathfrak{A}, \beta \frac{a}{x} \right) \models \psi \\ &\iff \text{there exists an } a \in A \text{ such that } \left(\mathfrak{J} \frac{a}{x} \right)^\pi = \left(\mathfrak{A}, \beta \frac{a}{x} \right)^\pi \models \psi, \\ &\quad \text{(by induction hypothesis on } \mathfrak{J} \frac{a}{x}, \left(\mathfrak{J} \frac{a}{x} \right)^\pi, \text{ and } \psi) \\ &\quad \text{that is, there exists an } a \in A \text{ such that } \left(\mathfrak{B}, \beta^\pi \frac{\pi(a)}{x} \right) \models \psi \\ &\iff \text{there exists a } b \in B \text{ such that } \left(\mathfrak{B}, \beta^\pi \frac{b}{x} \right) \models \psi \quad \text{(since } \pi \text{ is surjective)} \\ &\quad \text{i.e., there exists a } b \in B \text{ with } \mathfrak{J}^\pi \frac{b}{x} = \left(\mathfrak{B}, \beta^\pi \frac{b}{x} \right) \models \psi \\ &\iff \mathfrak{J}^\pi \models \exists x\psi. \end{aligned}$$

This finishes the proof. □

Corollary 1.6. Let $\pi : \mathfrak{A} \cong \mathfrak{B}$ and $\varphi \in L_n^S$. Then for every a_0, \dots, a_{n-1}

$$\mathfrak{A} \models \varphi[a_0, \dots, a_{n-1}] \iff \mathfrak{B} \models \varphi[\pi(a_0), \dots, \pi(a_{n-1})]$$

⊢

2. Exercises

Exercise 2.1. Prove Lemma 1.4.

Exercise 2.2. Let φ , ψ , and χ be S -formulas. Prove that:

- (a) $(\varphi \vee \psi) \models \chi$ if and only if $\varphi \models \chi$ and $\psi \models \chi$.
- (b) $\models \varphi \rightarrow \psi$ if and only if $\varphi \models \psi$.

Exercise 2.3. Let S be finite, i.e., containing finitely many relation symbols, function symbols, and constants. Prove that two *finite* structures \mathfrak{A} and \mathfrak{B} are isomorphic if and only if for any S -sentence φ

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi.$$