Mathematical Logic (V)

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1. Sequent Calculus

1.1. Basic Rules.

Antecedent.

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi}\ \Gamma \subseteq \Gamma'$$

The correctness is straightforward. Assume that $\Gamma \models \varphi$ and $\mathfrak{I} \models \Gamma'$. Since $\Gamma \subseteq \Gamma'$, we conclude $\mathfrak{I} \models \Gamma$ and thus $\mathfrak{I} \models \varphi$.

Assumption.

$$\frac{}{\Gamma \quad \varphi}\ \varphi \in \Gamma$$

Case Analysis.

$$\frac{\begin{array}{ll}\Gamma\ \psi & \varphi \\ \Gamma\ \neg\psi & \varphi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

Contradiction.

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

$\vee$*-introduction in antecedent.*

$$\frac{\begin{array}{lll}\Gamma & \varphi & \chi \\ \Gamma & \psi & \chi\end{array}}{\begin{array}{lll}\Gamma & (\varphi \vee \psi) & \chi\end{array}}$$

$\vee$*-introduction in consequence.*

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)} \qquad\qquad \text{(b)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$$

$\exists$*-introduction in consequence.*

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad \exists x\varphi}$$

$\exists$*-introduction in antecedent.*

$$\frac{\Gamma \quad \varphi\frac{y}{x} \quad \psi}{\Gamma \quad \exists x\varphi \quad \psi}\ \text{if } y \notin \text{free}\big(\Gamma \cup \{\exists x\varphi, \psi\}\big)$$

Equality.

$$\frac{}{t \equiv t}$$

Substitution.

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}}$$

1.2. Some Derived Rules.

Example 1.1 (The law of excluded middle).

1.	φ	φ	(assumption)
2.	φ	$(\varphi \vee \neg\varphi)$	($\vee$-introduction in consequence by 1)
3.	$\neg\varphi$	$\neg\varphi$	(assumption)
4.	$\neg\varphi$	$(\varphi \vee \neg\varphi)$	($\vee$-introduction in consequence by 3)
5.		$(\varphi \vee \neg\varphi)$	(case analysis by 2 and 4).

Therefore $\vdash (\varphi \vee \neg\varphi)$. ⊣

Example 1.2 (The modified contradiction).

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

We argue as follows.

1.	Γ	ψ	(premise)
2.	Γ	$\neg\psi$	(premise)
3.	$\Gamma \ \neg\varphi$	ψ	(antecedent by 1)
4.	$\Gamma \ \neg\varphi$	$\neg\psi$	(antecedent by 2)
5.	Γ	φ	(contradiction by 3 and 4).

⊣

Example 1.3 (The chain deduction).

$$\frac{\begin{array}{rl}\Gamma & \varphi \\ \Gamma \ \varphi & \psi\end{array}}{\begin{array}{rl}\Gamma & \psi\end{array}}$$

We have the following deduction.

1.	Γ	φ	(premise)
2.	$\Gamma \ \varphi$	ψ	(premise)
3.	$\Gamma \ \neg\varphi$	φ	(antecedent by 1)
4.	$\Gamma \ \neg\varphi$	$\neg\varphi$	(assumption)
5.	$\Gamma \ \neg\varphi$	ψ	(modified contradiction by 3 and 4)
6.	Γ	ψ	(case analysis by 2 and 5).

⊣

Definition 1.4. Let Φ be a set of S-formulas and φ an S-formula. Then φ is *derivable from* Φ, denoted by $\Phi \vdash \varphi$, if there exists an $n \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_n \in \Phi$ such that

$$\vdash \varphi_1 \ldots \varphi_n \ \varphi.$$ ⊣

Let Φ be a set of S-sentences and φ an S-formula.

Lemma 1.5. $\Phi \vdash \varphi$ *if and only if there exists a* finite $\Phi_0 \subseteq \Phi$ *such that* $\Phi_0 \vdash \varphi$. $\dashv$

Theorem 1.6 (Soundness). *If* $\Phi \vdash \varphi$, *then* $\Phi \models \varphi$. $\dashv$

2. Consistency

Definition 2.1. Φ is *consistent*, written $\mathrm{cons}(\Phi)$, if there is no φ such that both $\Phi \vdash \varphi$ and $\Phi \vdash \varphi$. Otherwise, Φ is *inconsistent*.

Lemma 2.2. Φ *is inconsistent if and only if* $\Phi \vdash \varphi$ *for any formula* φ.

Proof: The direction from right to left is by Definition 2.1. For the converse direction, assume that there is a ψ such that $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$. Then there exist two finite sequences of formulas, Γ_1 and Γ_2, such that we have derivation

$$\begin{array}{c} \vdots \\ \Gamma_1 \quad \psi \end{array} \qquad \text{and} \qquad \begin{array}{c} \vdots \\ \Gamma_2 \quad \neg\psi. \end{array}$$

Then for every φ we can obtain the derivation of $\Gamma_1\ \Gamma_2\ \varphi$ as below.

$$\begin{array}{rccrr}
 & & \vdots & & \\
m. & & \Gamma_1 & \psi & \\
 & & \vdots & & \\
n. & & \Gamma_2 & \neg\psi & \\
(n+1). & \Gamma_1 & \Gamma_2 & \psi & \text{(antecedent by } m) \\
(n+2). & \Gamma_1 & \Gamma_2 & \neg\psi & \text{(antecedent by } n) \\
(n+3). & \Gamma_1 & \Gamma_2 & \varphi & \text{(modified contradiction by } n+1 \text{ and } n+2).
\end{array}$$

□

Corollary 2.3. Φ *is consistent if and only if there is a* φ *such that* $\Phi \nvdash \varphi$.

Lemma 2.4. Φ *is consistent if and only if every finite* $\Phi_0 \subseteq \Phi$ *is consistent.*

Lemma 2.5. *Every satisfiable* Φ *is consistent.*

Proof: Assume that Φ is inconsistent. Then there is a φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. By the Soundness Theorem, i.e., Theorem 1.6, we conclude $\Phi \models \varphi$ and $\Phi \models \neg\varphi$. Thus, Φ cannot be satisfiable. □

Lemma 2.6. *(a)* $\Phi \vdash \varphi$ *if and only if* $\Phi \cup \{\neg\varphi\}$ *is inconsistent.*

(b) $\Phi \vdash \neg\varphi$ *if and only if* $\Phi \cup \{\varphi\}$ *is inconsistent.*

(c) If $\mathrm{cons}(\Phi)$, *then either* $\mathrm{cons}\big(\Phi \cup \{\varphi\}\big)$ *or* $\mathrm{cons}(\Phi \cup \{\neg\varphi\})$.

3. Exercises

Exercise 3.1. Can you derive the rule of contradiction from the modified contradiction?

Exercise 3.2. Prove:

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad \neg\neg\varphi} \qquad\qquad \text{(b)}\ \frac{\Gamma \quad \neg\neg\varphi}{\Gamma \quad \varphi}$$

Exercise 3.3. Is the following derivable?

$$\frac{}{\Gamma \quad \exists x\varphi \qquad \forall x\varphi}$$