Mathematical Logic (VI)

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1. Completeness

The goal of this section is to show:

Theorem 1.1 (Completeness). *If* $\Phi \models \varphi$*, then* $\Phi \vdash \varphi$. ⊣

We observe that the contrapositive of Theorem 1.1 is:

$$\begin{aligned} & \Phi \nvdash \varphi \text{ implies } \Phi \not\models \varphi \\ \iff & \text{if } \Phi \cup \{\neg\varphi\} \text{ is consistent, then } \Phi \cup \{\neg\varphi\} \text{ is satisfiable.} \end{aligned}$$

As a matter of fact, we actually will prove the following general statement.

Theorem 1.2. $\mathrm{cons}(\Phi)$ *implies that* Φ *is satisfiable.* ⊣

1.1. Henkin's Theorem.

We fix a set Φ of S-formulas and will construct an S-interpretation out of Φ. To that end, we first define a binary relation over the set T^S of S-terms.

Definition 1.3. Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. ⊣

Lemma 1.4. *(i)* $\sim$ *is an* equivalence *relation.*

(ii) $\sim$ *is a* congruence *relation. That is:*

- *For every* n*-ary function symbol* $f \in S$ *and* $2 \cdot n$ S*-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$ft_1 \cdots t_n \sim ft_1' \cdots t_n'.$$

- *For every* n*-ary relation symbol* $R \in S$ *and* $2 \cdot n$ S*-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$\Phi \vdash Rt_1 \cdots t_n \iff \Phi \vdash Rt_1' \cdots t_n'.$$

⊣

Proof: By the equality rule and the substitution rule. □

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of t.

Definition 1.5. The *term structure for* Φ, denoted by $\mathfrak{T}^\Phi$, is defined as follows.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

(ii) For every n-ary relation symbol $R \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \ldots, \bar{t}_n) \in R^{\mathfrak{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \ldots t_n.$$

(iii) For every n-ary function symbol $f \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) := \overline{ft_1 \cdots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathfrak{T}^\Phi} := \bar{c}.$$

This finishes the construction of $\mathfrak{T}^\Phi$. $\dashv$

Using Lemma 1.4, in particular (ii), it is easy to verify that:

Lemma 1.6. $\mathfrak{T}^\Phi$ *is well-defined.* $\dashv$

To complete the definition of an S-interpretation, we still need to provide an assignment of the variables $v_0, v_1, \ldots$ in $\mathfrak{T}^\Phi$.

Definition 1.7. For every variable v_i we let

$$\beta^\Phi(v_i) := \bar{v}_i. \qquad \dashv$$

Finally we let

$$\mathfrak{I}^\Phi := \left(\mathfrak{T}^\Phi, \beta^\Phi\right).$$

Lemma 1.8. *(i) For any* $t \in T^S$

$$\mathfrak{I}^\Phi(t) = \bar{t}.$$

(ii) For every atomic φ

$$\mathfrak{T}^\Phi \models \varphi \quad \Longleftrightarrow \quad \Phi \vdash \varphi.$$

Proof: (i) We proceed by induction on t.

- $t = v_i$ is a variable. Then
$$\mathfrak{I}^\Phi(v_i) = \beta^\Phi(v_i) = \bar{v}_i.$$
- $t = c$ is a constant. Then
$$\mathfrak{I}^\Phi(c) = c^{\mathfrak{T}^\Phi} = \bar{c}$$
- $t = ft_1 \cdots t_n$. Then
$$\begin{aligned}\mathfrak{I}^\Phi(ft_1 \cdots t_n) &= f^{\mathfrak{T}^\Phi}(\mathfrak{I}^\Phi(t_1), \ldots, \mathfrak{I}^\Phi(t_n)) \\ &= f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) && \text{(by induction hypothesis)} \\ &= \overline{ft_1 \cdots t_n}.\end{aligned}$$

(ii) Recall that there are two types of atomic formulas. For the first type, let $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned}\mathfrak{T}^\Phi \models t_1 \equiv t_2 &\iff \mathfrak{I}^\Phi(t_1) = \mathfrak{I}^\Phi(t_2) \\ &\iff \bar{t}_1 = \bar{t}_2 && \text{(by (i))} \\ &\iff t_1 \sim t_2 \\ &\iff \Phi \vdash t_1 \equiv t_2.\end{aligned}$$

Second, let $\varphi = Rt_1 \cdots t_n$. We deduce

$$\begin{aligned}\mathfrak{T}^\Phi \models Rt_1 \cdots t_n &\iff \left(\mathfrak{I}^\Phi(t_1), \ldots, \mathfrak{I}^\Phi(t_n)\right) \in R^{\mathfrak{T}^\Phi} \\ &\iff \left(\bar{t}_1, \ldots, \bar{t}_n\right) \in R^{\mathfrak{T}^\Phi} && \text{(by (i))} \\ &\iff \Phi \vdash Rt_1 \cdots t_n.\end{aligned}$$

□

2. Exercises

Exercise 2.1. Assume that Φ is inconsistent. Please describe the structure $\mathfrak{T}^{\Phi}$.

Exercise 2.2. Let $S := \{R\}$ with unary relation symbol R. Moreover we define

$$\Phi := \{\exists x R x\} \cup \{\neg R y \mid \text{for every variable } y\}.$$

Prove that:

- Φ is consistent.
- There is no term $t \in T^S$ with $\Phi \vdash Rt$.

Exercise 2.3. Again let $S := \{R\}$ with unary relation symbol R, and

$$\Phi := \{Rx \vee Ry\}.$$

Prove that:

- Φ is consistent.
- $\Phi \nvdash Rx$ and $\Phi \nvdash Ry$.
- $\mathfrak{T}^{\Phi} \not\models \Phi$.