

Mathematical Logic (VII)

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1. Completeness

1.1. Henkin's Theorem. Recall that we fix a set Φ of S -formulas.

Definition 1.1. Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. +

Lemma 1.2. (i) \sim is an equivalence relation.

(ii) \sim is a congruence relation. That is:

– For every n -ary function symbol $R \in S$ and $2 \cdot n$ S -terms $t_1 \sim t'_1, \dots, t_n \sim t'_n$, we have

$$ft_1 \dots t_n \sim ft'_1 \dots t'_n.$$

– For every n -ary relation symbol $R \in S$ and $2 \cdot n$ S -terms $t_1 \sim t'_1, \dots, t_n \sim t'_n$, we have

$$\Phi \vdash Rt_1 \dots t_n \iff \Phi \vdash Rt'_1 \dots t'_n. \quad +$$

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of t .

Definition 1.3. The *term structure* for Φ , denoted by \mathfrak{T}^Φ , is defined as below.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

(ii) For every n -ary relation symbol $R \in S$, and $\bar{t}_1, \dots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \dots, \bar{t}_n) \in R^{\mathfrak{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \dots t_n.$$

(iii) For every n -ary function symbol $f \in S$, and $\bar{t}_1, \dots, \bar{t}_n \in T^\Phi$

$$f^{\mathfrak{T}^\Phi}(\bar{t}_1, \dots, \bar{t}_n) := \overline{ft_1 \dots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathfrak{T}^\Phi} := \bar{c}.$$

This finishes the construction of \mathfrak{T}^Φ . For every variable v_i let

$$\beta^\Phi(v_i) := \bar{v}_i.$$

Finally we set

$$\mathfrak{J}^\Phi := (\mathfrak{T}^\Phi, \beta^\Phi),$$

which is the *term interpretation* of Φ . +

Lemma 1.4. (i) For any $t \in T^S$

$$\mathcal{J}^\Phi(t) = \bar{t}.$$

(ii) For every atomic φ

$$\mathcal{J}^\Phi \models \varphi \iff \Phi \vdash \varphi. \quad \dashv$$

Lemma 1.5. Let φ be an S-formula and x_1, \dots, x_n pairwise distinct variables. Then

(i) $\mathcal{J}^\Phi \models \exists x_1 \dots \exists x_n \varphi$ if and only if there are S-terms t_1, \dots, t_n such that

$$\mathcal{J}^\Phi \models \varphi \frac{t_1 \dots t_n}{x_1 \dots x_n}.$$

(ii) $\mathcal{J}^\Phi \models \forall x_1 \dots \forall x_n \varphi$ if and only if for all S-terms t_1, \dots, t_n we have

$$\mathcal{J}^\Phi \models \varphi \frac{t_1 \dots t_n}{x_1 \dots x_n}.$$

Proof: We prove (i), then (ii) follows immediately.

$$\mathcal{J}^\Phi \models \exists x_1 \dots \exists x_n \varphi$$

$$\iff \mathcal{J}^\Phi \frac{a_1 \dots a_n}{x_1 \dots x_n} \models \varphi \text{ for some } a_1, \dots, a_n \in T^\Phi,$$

$$\text{i.e., } \mathcal{J}^\Phi \frac{\bar{t}_1 \dots \bar{t}_n}{x_1 \dots x_n} \models \varphi \text{ for some } t_1, \dots, t_n \in T^S,$$

$$\iff \mathcal{J}^\Phi \frac{\mathcal{J}^\Phi(t_1) \dots \mathcal{J}^\Phi(t_n)}{x_1 \dots x_n} \models \varphi \text{ for some } t_1, \dots, t_n \in T^S, \quad (\text{by Lemma 1.4 (i)})$$

$$\iff \mathcal{J}^\Phi \models \varphi \frac{t_1 \dots t_n}{x_1 \dots x_n} \text{ for some } t_1, \dots, t_n \in T^S, \quad (\text{by the Substitution Lemma}).$$

□

Definition 1.6. (i) Φ is *negation complete* if for every S-formula φ

$$\Phi \vdash \varphi \quad \text{or} \quad \Phi \vdash \neg \varphi.$$

(ii) Φ contains witnesses if for every S-formula φ and every variable x there is a term $t \in T^S$ with

$$\Phi \vdash \left(\exists x \varphi \rightarrow \varphi \frac{t}{x} \right). \quad \dashv$$

Lemma 1.7. Assume that Φ is consistent, negation complete, and contains witnesses. Then for all S-formulas φ and ψ :

(i) $\Phi \vdash \varphi$ if and only if $\Phi \not\vdash \neg \varphi$.

(ii) $\Phi \vdash (\varphi \vee \psi)$ if and only if $\Phi \vdash \varphi$ or $\Phi \vdash \psi$.

(iii) $\Phi \vdash \exists x \varphi$ if and only if there is a term $t \in T^S$ such that $\Phi \vdash \varphi \frac{t}{x}$.

Proof: (i) Assume that $\Phi \vdash \varphi$. Since Φ is consistent, we conclude that $\Phi \not\vdash \neg \varphi$. Conversely, if $\Phi \not\vdash \neg \varphi$, then $\Phi \vdash \varphi$ by the negation completeness.

(ii) The direction from right to left is trivial by \vee -introduction in consequence. For the other direction, assume that $\Phi \vdash (\varphi \vee \psi)$ and $\Phi \not\vdash \varphi$. By the negation completeness, $\Phi \vdash \neg \varphi$. Then for some finite $\Gamma \subseteq \Phi$ we have the following sequent proof.

m.	\vdots	Γ_1	$(\varphi \vee \psi)$	
	\vdots			
	\vdots	Γ_2	$\neg\varphi$	
(n + 1).	Γ_1	Γ_2	φ	$\neg\varphi$ (antecedent by m)
(n + 2).	Γ_1	Γ_2	φ	φ (assumption)
(n + 3).	Γ_1	Γ_2	φ	ψ (modified contradiction by n + 1 and n + 2)
(n + 4).	Γ_1	Γ_2	ψ	ψ (assumption)
(n + 5).	Γ_1	Γ_2	$(\varphi \vee \psi)$	ψ (V-introduction in antecedent)
(n + 6).	Γ_1	Γ_2	ψ	ψ (chain rule by m and n + 5)

(iii) Let $\Phi \vdash \exists x\varphi$ and Φ contain witnesses. Thus there is a term $t \in T^S$ such that

$$\Phi \vdash \left(\exists x\varphi \rightarrow \varphi \frac{t}{x} \right).$$

By Modus ponens, we conclude $\Phi \vdash \varphi \frac{t}{x}$. The converse is by the rule of the \exists -introduction in consequence. \square

Theorem 1.8 (Henkin's Theorem). *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then for every S-formula φ*

$$\mathcal{J}^\Phi \models \varphi \iff \Phi \vdash \varphi.$$

Proof: We proceed by induction on φ .

- φ is atomic. This is Lemma 1.4 (ii).
- $\varphi = \neg\psi$. Then

$$\begin{aligned} \mathcal{J}^\Phi \models \neg\psi &\iff \mathcal{J}^\Phi \not\models \psi \\ &\iff \Phi \not\vdash \psi && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash \neg\psi && \text{(by Lemma 1.7 (i)).} \end{aligned}$$

- $\varphi = (\psi_1 \vee \psi_2)$. We deduce

$$\begin{aligned} \mathcal{J}^\Phi \models (\psi_1 \vee \psi_2) &\iff \mathcal{J}^\Phi \models \psi_1 \text{ or } \mathcal{J}^\Phi \models \psi_2 \\ &\iff \Phi \vdash \psi_1 \text{ or } \Phi \vdash \psi_2 && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash (\psi_1 \vee \psi_2) && \text{(by Lemma 1.7 (ii)).} \end{aligned}$$

- $\varphi = \exists x\psi$.

$$\begin{aligned} \mathcal{J}^\Phi \models \exists x\psi &\iff \mathcal{J}^\Phi \models \psi \frac{t}{x} \text{ for some } t \in T^S && \text{(by Lemma 1.5)} \\ &\iff \Phi \vdash \psi \frac{t}{x} \text{ for some } t \in T^S && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash \exists x\psi && \text{(by Lemma 1.7 (iii)).} \end{aligned}$$

Here, note that the length of $\psi \frac{t}{x}$ could be well larger than that $\exists x\psi$. Thus, our induction is on the so-called *connectivity rank* of ψ , denoted by $\text{rk}(\varphi)$, which is defined as follows:

$$\text{rk}(\varphi) := \begin{cases} 0 & \text{if } \varphi \text{ is atomic,} \\ 1 + \text{rk}(\psi) & \text{if } \varphi = \neg\psi, \\ 1 + \text{rk}(\psi_1) + \text{rk}(\psi_2) & \text{if } \varphi = (\psi_1 \vee \psi_2), \\ 1 + \text{rk}(\psi) & \text{if } \varphi = \exists x\psi. \end{cases}$$

□

Corollary 1.9. *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then*

$$\mathcal{J}^\Phi \models \Phi.$$

In particular, Φ is satisfiable.

1.2. The countable case. We fix a symbol set S which is at most countable. As a consequence, both T^S and L^S are countable. Let $\Phi \subseteq L^S$ we define

$$\text{free}(\Phi) := \bigcup_{\varphi \in \Phi} \text{free}(\varphi).$$

We will prove the following two lemmas.

Lemma 1.10. *Let $\Phi \subseteq L^S$ be consistent with finite $\text{free}(\Phi)$. Then there is a consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ such that Ψ contains witnesses.*

Lemma 1.11. *Let $\Psi \subseteq L^S$ be consistent. Then there is a consistent Θ with $\Psi \subseteq \Theta \subseteq L^S$ such that Θ is negation complete.*

Corollary 1.12. *Let $\Phi \subseteq L^S$ be consistent with finite $\text{free}(\Phi)$. Then Φ is satisfiable.*

Proof: By Lemma 1.10, there is a consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ such that Ψ contains witnesses. Observe that any Θ with $\Psi \subseteq \Theta \subseteq L^S$ still contains witnesses. By Lemma 1.11 we can choose such a Θ which is negation complete. Now Corollary 1.9 implies that Θ is satisfiable. Since $\Phi \subseteq \Theta$, Φ is satisfiable as well. □

Proof of Lemma 1.10: Recall L^S is countable, thus we can enumerate all S -formulas

$$\exists x_0 \varphi_0, \exists x_1 \varphi_1, \dots,$$

which start with an existential quantifier. Then we define inductively for every $n \in \mathbb{N}$ an S -formula ψ_n as follows. Assume that ψ_m has been defined for all $m < n$. Let

$$i_n := \min\{i \in \mathbb{N} \mid v_i \notin \text{free}(\Phi \cup \{\psi_m \mid m < n\} \cup \{\exists x_n \varphi_n\})\}.$$

That is, i_n is the smallest index i such that v_i is not free in $\Phi \cup \{\psi_m \mid m < n\} \cup \{\exists x_n \varphi_n\}$. Then we set

$$\psi_n := \left(\exists x_n \varphi_n \rightarrow \varphi_n \frac{v_{i_n}}{x_n} \right).$$

Next, let

$$\Phi_n := \Phi \cup \{\psi_m \mid m < n\},$$

and $\Phi := \bigcup_{n \in \mathbb{N}} \Phi_n$. It should be clear that Φ contains witness. So what remains is to show that Φ is consistent, or equivalently every Φ_n is consistent.

Recall that $\Phi_0 = \Phi$ is consistent by our assumption. Towards a contradiction, assume that Φ_n is consistent, but Φ_{n+1} is not. Therefore, for every χ with $v_{i_n} \notin \text{free}(\exists x_n \varphi_n)$ there is a finite $\Gamma \subseteq \Phi_n$ with the following deduction.

		⋮			
	m.	$\Gamma \left(\neg \exists x_n \varphi_n \vee \varphi_n \frac{v_{i_n}}{x_n} \right)$	χ		
(m + 1).		$\Gamma \neg \exists x_n \varphi_n$	$\neg \exists x_n \varphi_n$	(assumption)	
(m + 2).		$\Gamma \neg \exists x_n \varphi_n$	$\left(\neg \exists x_n \varphi_n \vee \varphi_n \frac{v_{i_n}}{x_n} \right)$	(V-introduction in the consequence)	
(m + 3).		$\Gamma \neg \exists x_n \varphi_n$	χ	(chain rule)	
(m + 4).		$\Gamma \varphi_n \frac{v_{i_n}}{x_n}$	χ	(similarly)	
(n + 5).		$\Gamma \exists x_n \varphi_n$	χ	(\exists -introduction in the antecedent)	
(m + 6).		Γ	χ	(case analysis).	

Now by taking $\chi := \exists v_0 v_0 \equiv v_0$ and $\chi := \neg \exists v_0 v_0 \equiv v_0$ we conclude that Φ_n is inconsistent, which contradicts our assumption. \square

Proof of Lemma 1.11: Let $\varphi_0, \varphi_1, \dots$ be an enumeration of L^S . For every $n \in \mathbb{N}$ we define Θ_n by induction. First $\Theta_0 := \Psi$. Then,

$$\Theta_{n+1} := \begin{cases} \Theta_n \cup \{\varphi_n\} & \text{if } \Theta_n \cup \{\varphi_n\} \text{ is consistent,} \\ \Theta_n & \text{otherwise.} \end{cases}$$

It is immediate that every Θ_n is consistent, and the consistency of

$$\Theta := \bigcup_{n \in \mathbb{N}} \Theta_n$$

follows. To see that Θ is negation complete, let $\varphi \in L^S$, in particular $\varphi = \varphi_n$ for some $n \in \mathbb{N}$. Assuming $\Theta \not\vdash \neg \varphi_n$, we conclude $\Theta_n \not\vdash \neg \varphi_n$ by $\Theta_n \subseteq \Theta$. Therefore, $\Theta_n \cup \{\varphi\}$ is consistent. It follows that $\varphi \in \Theta_{n+1} \subseteq \Theta$, and thus $\Theta \vdash \varphi$. \square

2. Exercises

The exercise below shows that the finiteness of $\text{free}(\Phi)$ is necessary in Lemma 1.10.

Exercise 2.1. Let

$$\Phi := \{v_0 \equiv t \mid t \in T^S\} \cup \{\exists v_0 \exists v_1 \neg v_0 \equiv v_1\}.$$

Prove that Φ is consistent, but there is no consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ which contains witnesses. \dashv

Exercise 2.2. Let \mathfrak{A} and \mathfrak{B} be two S -structures. A mapping $h : A \rightarrow B$ is a *homomorphism* from \mathfrak{A} to \mathfrak{B} if the following properties hold.

(1) For every n -ary relation symbol $R \in S$ and $a_1, \dots, a_n \in A$ we have

$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \text{ implies } (h(a_1), \dots, h(a_n)) \in R^{\mathfrak{B}}.$$

(2) For every n -ary function symbol $R \in S$ and $a_1, \dots, a_n \in A$ we have

$$h(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)).$$

(3) For every constant $c \in S$

$$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

Now let $\Phi \subseteq L^S$ and \mathfrak{A} be an S -structure with $\mathfrak{A} \models \Phi$. Prove that there is a homomorphism from the term model \mathfrak{T}^Φ to \mathfrak{A} . \dashv