Mathematical Logic (III)

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1. The Semantics of First-order Logic

Lemma 1.1 (The Coincidence Lemma). *For* $i \in \{1, 2\}$ *let* $\mathfrak{I}_i = (\mathfrak{A}_i, \beta_i)$ *be an* S_i*-interpretation such that* $A_1 = A_2$ *and every symbol in* $S := S_1 \cap S_2$ *has the same interpretation in* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$*.*

(a) Let t *be an S-term (thus also an* S_1*-term and an* S_2*-term). Assume further that* $\beta_1(x) = \beta_2(x)$ *for every variable* $x \in \text{var}(t)$*. Then* $\mathfrak{I}_1(t) = \mathfrak{I}_2(t)$*.*

(b) Let φ *be an S-formula where* $\beta_1(x) = \beta_2(x)$ *for every* $x \in \text{free}(\varphi)$*. Then*

$$\mathfrak{I}_1 \models \varphi \quad \Longleftrightarrow \quad \mathfrak{I}_2 \models \varphi.$$

$\dashv$

Proof: (a) We prove by induction on t.

- $t = x$. Then $\mathfrak{I}_1(x) = \beta_1(x) = \beta_2(x) = \mathfrak{I}_2(x)$.
- $t = c$. We deduce $\mathfrak{I}_1(c) = c^{\mathfrak{A}_1} = c^{\mathfrak{A}_2} = \mathfrak{I}_2(x)$.
- $t = ft_1 \cdots t_n$. It holds that

$$\begin{aligned}\mathfrak{I}_1(ft_1 \cdots t_n) &= f^{\mathfrak{A}_1}\big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_2(t_n)\big) \\ &= f^{\mathfrak{A}_2}\big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_1(t_n)\big) \\ &= f^{\mathfrak{A}_2}\big(\mathfrak{I}_2(t_1), \ldots, \mathfrak{I}_2(t_n)\big) \\ &= \mathfrak{I}_2(ft_1 \cdots t_n).\end{aligned}$$

(b) The induction proof is on the structure of φ.

- $\varphi = t_1 \equiv t_2$. We have

$$\begin{aligned}\mathfrak{I}_1 \models t_1 \equiv t_2 &\iff \mathfrak{I}_1(t_1) = \mathfrak{I}_1(t_2) \\ &\iff \mathfrak{I}_2(t_1) = \mathfrak{I}_2(t_2) && \text{(by (a))} \\ &\iff \mathfrak{I}_2 \models t_1 \equiv t_2.\end{aligned}$$

- $\varphi = Rt_1 \cdots t_n$. Then

$$\begin{aligned}\mathfrak{I}_1 \models Rt_1 \cdots t_n &\iff \big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_1(t_n)\big) \in R^{\mathfrak{A}_1} \\ &\iff \big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_1(t_n)\big) \in R^{\mathfrak{A}_2} \\ &\iff \big(\mathfrak{I}_2(t_1), \ldots, \mathfrak{I}_2(t_n)\big) \in R^{\mathfrak{A}_2} \\ &\iff \mathfrak{I}_2 \models Rt_1 \cdots t_n.\end{aligned}$$

- $\varphi = \neg\psi$. We conclude

$$\mathfrak{I}_1 \models \neg\psi \iff \mathfrak{I}_1 \not\models \psi \iff \mathfrak{I}_2 \not\models \psi \iff \mathfrak{I}_2 \models \neg\psi.$$

- $\varphi = (\psi \vee \chi)$.

$$\begin{aligned}\mathfrak{I}_1 \models (\psi \vee \chi) &\iff \mathfrak{I}_1 \models \psi \text{ or } \mathfrak{I}_1 \models \chi \\ &\iff \mathfrak{I}_2 \models \psi \text{ or } \mathfrak{I}_2 \models \chi \\ &\iff \mathfrak{I}_2 \models (\psi \vee \chi).\end{aligned}$$

- $\varphi = \exists x \psi$.

$$\begin{aligned}\mathfrak{I}_1 \models \exists x \psi &\iff \text{for some } a \in A_1 \text{ we have } \mathfrak{I}_1 \frac{a}{x} \models \psi \\ &\iff \text{for some } a \in A_1 \text{ we have } \mathfrak{I}_2 \frac{a}{x} \models \psi \\ &\qquad\qquad \Big(\text{by induction hypothesis on } \mathfrak{I}_1 \frac{a}{x}, \mathfrak{I}_2 \frac{a}{x}, \text{ and } \psi\Big) \\ &\iff \mathfrak{I}_2 \models \exists x \psi.\end{aligned}$$

□

Remark 1.2. Let $\varphi \in L_n^S$, i.e., φ is an S-formula with $\text{free}(\varphi) \subseteq \{v_0, \ldots, v_{n-1}\}$. By the coincidence lemma whether $\mathfrak{I} = (\mathfrak{A}, \beta) \models \varphi$ is completely determined by $\mathfrak{A}$ and $\beta(v_0), \ldots, \beta(v_{n-1})$. So in case $\mathfrak{I} \models \varphi$ we can write

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}]$$

where $a_i := \beta(v_i)$ for $0 \leqslant i < n$. In particular, if φ is an S-sentence, i.e., $\varphi \in L_0^S$, then $\mathfrak{A} \models \varphi$ is well-defined.

Similarly, we write

$$t^{\mathfrak{A}}[a_0, \ldots, a_{n-1}]$$

instead of $\mathfrak{I}(t)$. ⊣

Definition 1.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two S-structures.

(a) A mapping $\pi : A \to B$ is an *isomorphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$ (in short $\pi : \mathfrak{A} \cong \mathfrak{B}$) if the following conditions are satisfied.

 (i) π is a bijection.

 (ii) For any n-ary relation symbol $R \in S$ and $a_0, \ldots, a_{n-1} \in A$

$$(a_0, \ldots, a_{n-1}) \in R^{\mathfrak{A}} \iff \big(\pi(a_0), \ldots, \pi(a_{n-1})\big) \in R^{\mathfrak{B}}.$$

 (iii) For any n-ary function symbol $f \in S$ and $a_0, \ldots, a_{n-1} \in A$

$$\pi(f^{\mathfrak{A}}(a_0, \ldots, a_{n-1})) = f^{\mathfrak{B}}(\pi(a_0), \ldots, \pi(a_{n-1})).$$

 (iv) For any constant $c \in S$

$$\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

(b) $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, written $\mathfrak{A} \cong \mathfrak{B}$, if there is an isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$. ⊣

Observe that the above definition is not symmetric. However we can easily show:

Lemma 1.4. $\cong$ *is an equivalence relation. That is, for all* S*-structures* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$

(1) $\mathfrak{A} \cong \mathfrak{A}$;

(2) $\mathfrak{A} \cong \mathfrak{B}$ *implies* $\mathfrak{B} \cong \mathfrak{A}$;

(3) if $\mathfrak{A} \cong \mathfrak{B}$ *and* $\mathfrak{B} \cong \mathfrak{C}$*, then* $\mathfrak{A} \cong \mathfrak{C}$. ⊣

Lemma 1.5 (The Isomorphism Lemma). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two isomorphic S-structures. Then for every S-sentence* φ

$$\mathfrak{A} \models \varphi \quad \iff \quad \mathfrak{B} \models \varphi.$$

$\dashv$

Proof: Let β be an assignment in $\mathfrak{A}$. By the coincidence lemma, it suffices to show that there is an assignment β' in $\mathfrak{B}$ such that

$$(\mathfrak{A}, \beta) \models \varphi \quad \iff \quad (\mathfrak{B}, \beta') \models \varphi, \tag{1}$$

where φ is an S-sentence.

Let $\pi : \mathfrak{A} \cong \mathfrak{B}$ and we define an assignment β^π in $\mathfrak{B}$ by

$$\beta^\pi(x) := \pi(\beta(x))$$

for any variable x. Then we prove for any S-*formula* φ

$$(\mathfrak{A}, \beta) \models \varphi \quad \iff \quad (\mathfrak{B}, \beta^\pi) \models \varphi, \tag{2}$$

which certainly generalizes (1). To simplify notation, let $\mathfrak{I} := (\mathfrak{A}, \beta)$ and $\mathfrak{I}^\pi := \big(\mathfrak{B}, \beta^\pi\big)$. First, it is routine to verify that for every S-term t

$$\pi(\mathfrak{I}(t)) = \mathfrak{I}^\pi(t). \tag{3}$$

Then we prove (2) by induction on the construction of S-formula φ.

- $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned}
\mathfrak{I} \models t_1 \equiv t_2 &\iff \mathfrak{I}(t_1) = \mathfrak{I}(t_2) \\
&\iff \pi(\mathfrak{I}(t_1)) = \pi(\mathfrak{I}(t_2)) && \text{(since } \pi \text{ is an injection)} \\
&\iff \mathfrak{I}^\pi(t_1) = \mathfrak{I}^\pi(t_2) && \text{(by (3))} \\
&\iff \mathfrak{I}^\pi \models t_1 \equiv t_2.
\end{aligned}$$

- $\varphi = Rt_1 \cdots t_n$.

$$\begin{aligned}
\mathfrak{I} \models Rt_1 \cdots t_n &\iff \big(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)\big) \in R^{\mathfrak{A}} \\
&\iff \big(\pi(\mathfrak{I}(t_1)), \dots, \pi(\mathfrak{I}(t_n))\big) \in R^{\mathfrak{B}} \\
&\iff \big(\mathfrak{I}^\pi(t_1), \dots, \mathfrak{I}^\pi(t_n)\big) \in R^{\mathfrak{B}} && \text{(by (3))} \\
&\iff \mathfrak{I}^\pi \models Rt_1 \cdots t_n.
\end{aligned}$$

- $\varphi = \neg\psi$. It follows that $\mathfrak{I} \models \neg\psi \iff \mathfrak{I} \not\models \psi \iff \mathfrak{I}^\pi \not\models \iff \mathfrak{I}^\pi \models \neg\psi$.
- $\varphi = \psi \vee \chi$. The inductive argument is similar to the above $\neg\psi$.
- $\varphi = \exists x\psi$. This is again the most complicated case.

$$\begin{aligned}
\mathfrak{I} \models \exists x\psi &\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{a}{x} = \left(\mathfrak{A}, \beta\frac{a}{x}\right) \models \psi \\
&\iff \text{there exists an } a \in A \text{ such that } \left(\mathfrak{I}\frac{a}{x}\right)^\pi = \left(\mathfrak{A}, \beta\frac{a}{x}\right)^\pi \models \psi, \\
&\qquad\qquad \left(\text{by induction hypothesis on } \mathfrak{I}\frac{a}{x}, \left(\mathfrak{I}\frac{a}{x}\right)^\pi, \text{ and } \psi\right) \\
&\qquad \text{that is, there exists an } a \in A \text{ such that } \left(\mathfrak{B}, \beta^\pi\frac{\pi(a)}{x}\right) \models \psi \\
&\iff \text{there exists a } b \in B \text{ such that } \left(\mathfrak{B}, \beta^\pi\frac{b}{x}\right) \models \psi \qquad \text{(since } \pi \text{ is surjective)} \\
&\qquad \text{i.e., there exists a } b \in B \text{ with } \mathfrak{I}^\pi\frac{b}{x} = \big(\mathfrak{B}, \beta^\pi\big)\frac{b}{x} \models \psi \\
&\iff \mathfrak{I}^\pi \models \exists x\psi.
\end{aligned}$$

This finishes the proof. □

Corollary 1.6. *Let* $\pi : \mathfrak{A} \cong \mathfrak{B}$ *and* $\varphi \in L^S_n$*. Then for every* $a_0, \dots, a_{n-1}$

$$\mathfrak{A} \models \varphi[a_0, \dots, a_{n-1}] \quad \Longleftrightarrow \quad \mathfrak{B} \models \varphi\big[\pi(a_0), \dots, \pi(a_{n-1})\big]$$

⊣

2. Exercises

Exercise 2.1. Prove Lemma 1.4.

Exercise 2.2. Let φ, ψ, and χ be S-formulas. Prove that:

(a) $(\varphi \vee \psi) \models \chi$ if and only if $\varphi \models \psi$ and $\psi \models \chi$.

(b) $\models \varphi \to \psi$ if and only if $\varphi \models \psi$.

Exercise 2.3. Let S be finite, i.e., containing finitely many relation symbols, function symbols, and constants. Prove that two *finite* structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if and only if for any S-sentence φ

$$\mathfrak{A} \models \varphi \quad \Longleftrightarrow \quad \mathfrak{B} \models \varphi.$$