Mathematical Logic (IV)

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1. Substitution

In mathematics, when writing $f(y+10)$ we plug the value of $y+10$ into $f(x)$. We will do the same for $\varphi(x)$ where we want to substitute x by a term t. This is not completely trivial, e.g.,

$$\varphi(x) = \exists z\, z+z \equiv x \quad \text{and} \quad t = x+z.$$

It is obviously wrong for

$$\exists z\, z+z \equiv x+z.$$

Definition 1.1. Let t be an S-term, $x_0, \dots, x_r$ variables, and $t_0, \dots, t_r$ S-terms. Then the term

$$t\frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

is defined inductively as follows.

(a) Let $t = x$ be a variable. Then

$$t\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \begin{cases} t_i & \text{if } x = x_i \text{ for some } 0 \leqslant i \leqslant r \\ x & \text{otherwise.} \end{cases}$$

(b) For a constant $t = c$

$$c\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := c.$$

(c) For a function term

$$ft'_1 \dots t'_n\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := ft'_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$ ⊣

Definition 1.2. Let φ be an S-formula, $x_0, \dots, x_r$ variables, and $t_0, \dots, t_r$ S-terms. We define

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

inductively as follow.

(a) Assume $\varphi = t'_1 \equiv t'_2$. Then

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := t'_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \equiv t'_2\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(b) Let $\varphi = Rt'_1 \dots t'_n$. We set

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := Rt'_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(c) For $\varphi = \neg\psi$

$$\varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} := \neg\psi\frac{t_0,\dots,t_r}{x_0,\dots,x_r}.$$

(d) For $\varphi = (\psi_1 \vee \psi_2)$

$$\varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} := \left(\psi_1\frac{t_0,\dots,t_r}{x_0,\dots,x_r} \vee \psi_2\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right).$$

(e) Assume $\varphi = \exists x\psi$. Let $x_{i_1},\dots,x_{i_s}$ $(i_1 < \dots < i_s)$ be the variables x_i in $x_0,\dots,x_r$ with $x_i \in \text{free}(\exists x\varphi)$ and $x_i \neq t_i$. In particular, $x \neq x_{i_1},\dots,x \neq x_{i_s}$. Then

$$\varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} := \exists u\left[\psi\frac{t_{i_1},\dots,t_{i_s},u}{x_{i_1},\dots,x_{i_s},x}\right],$$

where $u = x$ if x does not occur in $t_{i_1},\dots,t_{t_s}$; otherwise u is the first variable in $\{v_0,v_1,v_2,\dots\}$ which does not occur in $\psi, t_{i_1},\dots,t_{i_s}$. ⊣

Examples 1.3. (1)

$$[Pv_0fv_1v_2]\frac{v_2,v_0,v_1}{v_1,v_2,v_3} = Pv_0fv_2v_0.$$

(2)

$$[\exists v_0\ Pv_0fv_1v_2]\frac{v_4,fv_1v_1}{v_0,v_2} = \exists v_0\left[Pv_0fv_1v_2\frac{fv_1v_1,v_0}{v_2,v_0}\right] = \exists v_0\ Pv_0fv_1fv_1v_1.$$

(3)

$$[\exists v_0\ Pv_0fv_1v_2]\frac{v_0,v_2,v_4}{v_1,v_2,v_0} = \exists v_3\left[Pv_0fv_1v_2\frac{v_0,v_3}{v_1,v_0}\right] = \exists v_3\ Pv_3fv_0v_2.$$ ⊣

Definition 1.4. Let β be an assignment in $\mathfrak{A}$ and $a_0,\dots,a_r \in A$. Then

$$\beta\frac{a_0,\dots,a_r}{x_0,\dots,x_r}$$

is an assignment in $\mathfrak{A}$ defined by

$$\beta\frac{a_0,\dots,a_r}{x_0,\dots,x_r} := \begin{cases} a_i & \text{if } y = x_i \text{ for } 0 \leqslant i \leqslant r \\ \beta(y) & \text{otherwise.} \end{cases}$$

For an S-interpretation $\mathfrak{I} = (\mathfrak{A},\beta)$ we let

$$\mathfrak{I}\frac{a_0,\dots,a_r}{x_0,\dots,x_r} := \left(\mathfrak{A},\beta\frac{a_0,\dots,a_r}{x_0,\dots,x_r}\right).$$ ⊣

Lemma 1.5 (The Substitution Lemma)**.** *(a) For every S-term* t

$$\mathfrak{I}\left(t\frac{t_0,\dots,t_r}{x_0,\dots x_r}\right) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots x_r}(t).$$

(b) For every S-formula φ

$$\mathfrak{I} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots x_r} \iff \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots x_r} \models \varphi.$$ ⊣

Proof: (a) Assume $t = x$. If $x \neq x_i$ for all $0 \leqslant i \leqslant r$, then

$$t\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} = x.$$

Therefore,

$$\mathfrak{I}\left(t\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}\right) = \mathfrak{I}(x) = \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r}(x) = \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r}(t).$$

Otherwise, $x = x_i$ for some $0 \leqslant i \leqslant r$. Then $t\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} = t_i$. It follows that

$$\mathfrak{I}\left(t\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}\right) = \mathfrak{I}(t_i) = \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r}(x_i) = \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r}(t).$$

The other cases of t can be shown similarly.

(b) Assume that $\varphi = Rt'_1 \ldots t'_n$. Then

$$\begin{aligned}
\mathfrak{I} \models \varphi\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} &\iff \left(\mathfrak{I}\Big(t'_1\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}\Big), \ldots, \mathfrak{I}\Big(t'_n\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}\Big)\right) \in R^{\mathfrak{A}} \\
&\iff \left(\mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r}(t'_1), \ldots, \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r}(t'_n)\right) \in R^{\mathfrak{A}} \qquad \text{(by (a))} \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r} \models Rt'_1 \ldots t'_n \\
&\qquad \text{i.e., } \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r} \models \varphi.
\end{aligned}$$

For another case, let $\varphi = \exists x\psi$. Again, let $x_{i_1}, \ldots, x_{i_s}$ be the variables x_i with $x_i \in \text{free}(\exists x\psi)$ and $x_i \neq t_i$. Choose u according to Definition 1.2 (e). In particular, u does not occur in $t_{i_1}, \ldots, t_{i_s}$. Then

$$\begin{aligned}
\mathfrak{I} \models \varphi\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} &\iff \mathfrak{I} \models \exists u\left[\psi\frac{t_{i_1}, \ldots, t_{i_s}, u}{x_{i_1}, \ldots, x_{i_s}, x}\right] \\
&\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{a}{u} \models \psi\frac{t_{i_1}, \ldots, t_{i_s}, u}{x_{i_1}, \ldots, x_{i_s}, x} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{a}{u}\right]\frac{\mathfrak{I}\frac{a}{u}(t_{i_1}), \ldots, \mathfrak{I}\frac{a}{u}(t_{i_s}), \mathfrak{I}\frac{a}{u}(u)}{x_{i_1}, \ldots, x_{i_s}, x} \models \psi \\
&\qquad \text{(by induction hypothesis)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{a}{u}\right]\frac{\mathfrak{I}(t_{i_1}), \ldots, \mathfrak{I}(t_{i_s}), a}{x_{i_1}, \ldots, x_{i_s}, x} \models \psi \\
&\qquad \text{(by the coincidence lemma and that } u \text{ does not occur in } t_{i_1}, \ldots t_{i_s}) \\
&\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{\mathfrak{I}(t_{i_1}), \ldots, \mathfrak{I}(t_{i_s}), a}{x_{i_1}, \ldots, x_{i_s}, x} \models \psi \\
&\qquad \text{(by (either } u = x \text{ or } u \text{ does not occur in } \psi) \text{ and the coincidence lemma)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{\mathfrak{I}(t_{i_1}), \ldots, \mathfrak{I}(t_{i_s})}{x_{i_1}, \ldots, x_{i_s}}\right]\frac{a}{x} \models \psi \\
&\qquad \text{(since } x \neq x_{i_1}, \ldots, x \neq x_{i_s}) \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_{i_1}), \ldots, \mathfrak{I}(t_{i_s})}{x_{i_1}, \ldots, x_{i_s}} \models \exists x\psi \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_0), \ldots, \mathfrak{I}(t_r)}{x_0, \ldots, x_r} \models \exists x\psi \\
&\qquad \text{(by } x_i \notin \text{free}(\exists x\psi) \text{ or } x_i = t_i \text{ for } i \neq i_1, \ldots, i \neq i_s).
\end{aligned}$$

□

2. Sequent Calculus

The goal of this section is to provide a formal definition of proofs, i.e., proofs are made into mathematical objects. To that end, we divide any proof into stages. In each stage, we establish a fact that under the *antecedent* $\varphi_1, \ldots, \varphi_n$[1] the *succedent* φ holds. In a succinct form, this is written as a sequent

$$\varphi_1 \ldots \varphi_n \varphi.$$

So our goal is to design a calculus $\mathfrak{S}$ operating on sequents, i.e., *sequent calculus*. $\mathfrak{S}$ contains a number of rules, which enable us to derive one sequent from another.

Definition 2.1. If in the calculus $\mathfrak{S}$ there is a derivation of the sequent $\Gamma\ \varphi$, then we write

$$\vdash \Gamma\ \varphi$$

and say that $\Gamma\ \varphi$ is *derivable*. $\dashv$

Definition 2.2. A formula φ is *formally provable* or *derivable* from a set Φ of formulas, written $\Phi \vdash \varphi$, if there are finite many formulas $\varphi_1, \ldots, \varphi_n$ in Φ such that

$$\vdash \varphi_1 \ldots \varphi_n\ \varphi. \qquad \dashv$$

Definition 2.3. A sequent $\Gamma\ \varphi$ is *correct* if

$$\{\psi \mid \psi \text{ is a member of } \Gamma\} \models \varphi.$$

in short, $\Gamma \models \varphi$. $\dashv$

2.1. Basic Rules.

Antecedent.

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi}\ \Gamma \subseteq \Gamma'$$

The correctness is straightforward. Assume that $\Gamma \models \varphi$ and $\mathfrak{I} \models \Gamma'$. Since $\Gamma \subseteq \Gamma'$, we conclude $\mathfrak{I} \models \Gamma$ and thus $\mathfrak{I} \models \varphi$.

Assumption.

$$\frac{}{\Gamma \quad \varphi}\ \varphi \in \Gamma$$

Case Analysis.

$$\frac{\begin{array}{lll}\Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

Contradiction.

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

$\vee$*-introduction in antecedent.*

[1] In the sequel, we tacitly assume a fixed symbol set S.

$$\frac{\begin{array}{rrr} & \Gamma\ \varphi & \chi \\ & \Gamma\ \psi & \chi \end{array}}{\Gamma\ (\varphi \vee \psi)\ \ \chi}$$

$\vee$*-introduction in succedent.*

$$\text{(a)}\ \frac{\Gamma\ \ \varphi}{\Gamma\ \ (\varphi \vee \psi)} \qquad \text{(b)}\ \frac{\Gamma\ \ \varphi}{\Gamma\ \ (\psi \vee \varphi)}$$

$\exists$*-introduction in succedent.*

$$\frac{\Gamma\ \ \varphi\frac{t}{x}}{\Gamma\ \ \exists x\varphi}$$

$\exists$*-introduction in antecedent.*

$$\frac{\Gamma\ \ \varphi\frac{y}{x}\ \ \psi}{\Gamma\ \ \exists x\varphi\ \ \psi}\ \text{if } y \notin \text{free}\big(\Gamma \cup \{\exists x\varphi, \psi\}\big)$$

Equality.

$$\overline{t \equiv t}$$

Substitution.

$$\frac{\Gamma\ \ \varphi\frac{t}{x}}{\Gamma\ \ t \equiv t'\ \ \varphi\frac{t'}{x}}$$

2.2. Some Derived Rules.

Example 2.4 (The law of excluded middle).

$$\begin{array}{lrrr}
1. & \varphi & \varphi & \text{(assumption)} \\
2. & \varphi & (\varphi \vee \neg\varphi) & \text{($\vee$-introduction in consequence by 1)} \\
3. & \neg\varphi & \neg\varphi & \text{(assumption)} \\
4. & \neg\varphi & (\varphi \vee \neg\varphi) & \text{($\vee$-introduction in consequence by 3)} \\
5. & & (\varphi \vee \neg\varphi) & \text{(case analysis by 2 and 4).}
\end{array}$$

Therefore $\vdash (\varphi \vee \neg\varphi)$. ⊣

Example 2.5 (The modified contradiction).

$$\frac{\begin{array}{ll} \Gamma & \psi \\ \Gamma & \neg\psi \end{array}}{\begin{array}{ll} \Gamma & \varphi \end{array}}$$

We argue as follows.

$$\begin{array}{lrrr}
1. & \Gamma & \psi & \text{(premise)} \\
2. & \Gamma & \neg\psi & \text{(premise)} \\
3. & \Gamma\ \neg\varphi & \psi & \text{(antecedent by 1)} \\
4. & \Gamma\ \neg\varphi & \neg\psi & \text{(antecedent by 2)} \\
5. & \Gamma & \varphi & \text{(contradiction by 3 and 4).}
\end{array}$$

⊣

Example 2.6 (The chain deduction).

$$\begin{array}{ccc} & \Gamma & \varphi \\ \Gamma & \varphi & \psi \\ \hline & \Gamma & \psi \end{array}$$

We have the following deduction.

1.		Γ	φ	(premise)
2.	Γ	φ	ψ	(premise)
3.	Γ	$\neg\varphi$	φ	(antecedent by 1)
4.	Γ	$\neg\varphi$	$\neg\varphi$	(assumption)
5.	Γ	$\neg\varphi$	ψ	(modified contradiction by 3 and 4)
6.		Γ	ψ	(case analysis by 2 and 5).

$\dashv$

Let Φ be a set of sentences and φ an formula.

Lemma 2.7. $\Phi \vdash \varphi$ *if and only if there exists a* finite $\Phi_0 \subseteq \Phi$ *such that* $\Phi_0 \vdash \varphi$. $\dashv$

Theorem 2.8 (Soundness). *If* $\Phi \vdash \varphi$*, then* $\Phi \models \varphi$. $\dashv$

3. Exercises

Exercise 3.1. Let P be a binary relation symbol and f a binary function symbol. Set $x := v_0$, $y := v_1$, $u := v_2$, $v := v_3$, and $w := v_4$. Show that:

(a)
$$\exists x \exists y \big(Pxu \wedge Pyv\big) \frac{u, u, u}{x, y, v} = \exists x \exists y \big(Pxu \wedge Pyu\big).$$

(b)
$$\exists x \exists y \big(Pxu \wedge Pyv\big) \frac{v, fuv}{u, v} = \exists x \exists y \big(Pxv \wedge Pyfuv\big).$$

(c)
$$\exists x \exists y \big(Pxu \wedge Pyv\big) \frac{u, x, fuv}{x, u, v} = \exists w \exists y \big(Pwx \wedge Pyfuv\big).$$

(c)
$$\big[\forall x \exists y (Pxy \wedge Pxu) vee \exists u\ fuu \equiv x\big] \frac{x, fxy}{x, u} = \forall v \exists w (Pvw \wedge Pvfxy) \vee \exists u fuu \equiv x.$$