Mathematical Logic (V)

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1. Sequent Calculus

1.1. Basic Rules.

Antecedent.

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi}\ \Gamma \subseteq \Gamma'$$

The correctness is straightforward. Assume that $\Gamma \models \varphi$ and $\mathfrak{I} \models \Gamma'$. Since $\Gamma \subseteq \Gamma'$, we conclude $\mathfrak{I} \models \Gamma$ and thus $\mathfrak{I} \models \varphi$.

Assumption.

$$\frac{}{\Gamma \quad \varphi}\ \varphi \in \Gamma$$

Case Analysis.

$$\frac{\begin{array}{ll}\Gamma\ \psi & \varphi \\ \Gamma\ \neg\psi & \varphi\end{array}}{\begin{array}{ll}\Gamma & \quad\ \varphi\end{array}}$$

Contradiction.

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \qquad\quad \varphi\end{array}}$$

$\vee$*-introduction in antecedent.*

$$\frac{\begin{array}{lll}\Gamma & \varphi & \chi \\ \Gamma & \psi & \chi\end{array}}{\begin{array}{lll}\Gamma & (\varphi \vee \psi) & \chi\end{array}}$$

$\vee$*-introduction in succedent.*

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)} \qquad\qquad \text{(b)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$$

$\exists$*-introduction in succedent.*

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad \exists x\varphi}$$

$\exists$*-introduction in antecedent.*

$$\frac{\Gamma \quad \varphi\frac{y}{x} \quad \psi}{\Gamma \quad \exists x\varphi \quad \psi}\ \text{if } y \notin \text{free}\big(\Gamma \cup \{\exists x\varphi, \psi\}\big)$$

Equality.

$$\frac{}{t \equiv t}$$

Substitution.

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}}$$

1.2. Some Derived Rules.

Example 1.1 (The law of excluded middle).

1.	φ	φ	(assumption)
2.	φ	$(\varphi \vee \neg\varphi)$	($\vee$-introduction in succedent by 1)
3.	$\neg\varphi$	$\neg\varphi$	(assumption)
4.	$\neg\varphi$	$(\varphi \vee \neg\varphi)$	($\vee$-introduction in succedent by 3)
5.		$(\varphi \vee \neg\varphi)$	(case analysis by 2 and 4).

Therefore $\vdash (\varphi \vee \neg\varphi)$. ⊣

Example 1.2 (The modified contradiction).

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

We argue as follows.

1.	Γ	ψ	(premise)
2.	Γ	$\neg\psi$	(premise)
3.	$\Gamma \quad \neg\varphi$	ψ	(antecedent by 1)
4.	$\Gamma \quad \neg\varphi$	$\neg\psi$	(antecedent by 2)
5.	Γ	φ	(contradiction by 3 and 4).

⊣

Example 1.3 (The chain deduction).

$$\frac{\begin{array}{rl}\Gamma & \varphi \\ \Gamma \quad \varphi & \psi\end{array}}{\begin{array}{rl}\Gamma & \psi\end{array}}$$

We have the following deduction.

1.	Γ	φ	(premise)
2.	$\Gamma \quad \varphi$	ψ	(premise)
3.	$\Gamma \quad \neg\varphi$	φ	(antecedent by 1)
4.	$\Gamma \quad \neg\varphi$	$\neg\varphi$	(assumption)
5.	$\Gamma \quad \neg\varphi$	ψ	(modified contradiction by 3 and 4)
6.	Γ	ψ	(case analysis by 2 and 5).

⊣

Definition 1.4. Let Φ be a set of S-formulas and φ an S-formula. Then φ is *derivable from* Φ, denoted by $\Phi \vdash \varphi$, if there exists an $n \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_n \in \Phi$ such that

$$\vdash \varphi_1 \ldots \varphi_n \ \varphi.$$ ⊣

Let Φ be a set of S-sentences and φ an S-formula.

Lemma 1.5. *$\Phi \vdash \varphi$ if and only if there exists a* finite *$\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vdash \varphi$.* $\dashv$

Theorem 1.6 (Soundness)**.** *If $\Phi \vdash \varphi$, then $\Phi \models \varphi$.* $\dashv$

2. Consistency

Definition 2.1. Φ is *consistent*, written $\mathrm{cons}(\Phi)$, if there is no φ such that both $\Phi \vdash \varphi$ and $\Phi \vdash \varphi$. Otherwise, Φ is *inconsistent*.

Lemma 2.2. *Φ is inconsistent if and only if $\Phi \vdash \varphi$ for any formula φ.*

Proof: The direction from right to left is by Definition 2.1. For the converse direction, assume that there is a ψ such that $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$. Then there exist two finite sequences of formulas, Γ_1 and Γ_2, such that we have derivation

$$\begin{array}{cc} & \vdots \\ \Gamma_1 & \psi \end{array} \quad \text{and} \quad \begin{array}{cc} & \vdots \\ \Gamma_2 & \neg\psi. \end{array}$$

Then for every φ we can obtain the derivation of $\Gamma_1\ \Gamma_2\ \varphi$ as below.

$$\begin{array}{rccrl}
 & & \vdots & & \\
m. & & \Gamma_1 & \psi & \\
 & & \vdots & & \\
n. & & \Gamma_2 & \neg\psi & \\
(n+1). & \Gamma_1 & \Gamma_2 & \psi & \text{(antecedent by } m\text{)} \\
(n+2). & \Gamma_1 & \Gamma_2 & \neg\psi & \text{(antecedent by } n\text{)} \\
(n+3). & \Gamma_1 & \Gamma_2 & \varphi & \text{(modified contradiction by } n+1 \text{ and } n+2\text{)}.
\end{array}$$

□

Corollary 2.3. *Φ is consistent if and only if there is a φ such that $\Phi \nvdash \varphi$.*

Lemma 2.4. *Φ is consistent if and only if every finite $\Phi_0 \subseteq \Phi$ is consistent.*

Lemma 2.5. *Every satisfiable Φ is consistent.*

Proof: Assume that Φ is inconsistent. Then there is a φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. By the Soundness Theorem, i.e., Theorem 1.6, we conclude $\Phi \models \varphi$ and $\Phi \models \neg\varphi$. Thus, Φ cannot be satisfiable. □

Lemma 2.6. *(a) $\Phi \vdash \varphi$ if and only if $\Phi \cup \{\neg\varphi\}$ is inconsistent.*

(b) $\Phi \vdash \neg\varphi$ if and only if $\Phi \cup \{\varphi\}$ is inconsistent.

(c) If $\mathrm{cons}(\Phi)$*, then either* $\mathrm{cons}\big(\Phi \cup \{\varphi\}\big)$ *or* $\mathrm{cons}(\Phi \cup \{\neg\varphi\})$.

3. Completeness

The goal of this section is to show:

Theorem 3.1 (Completeness). *If $\Phi \models \varphi$, then $\Phi \vdash \varphi$.* $\dashv$

We observe that the contrapositive of Theorem 3.1 is:

$$\Phi \nvdash \varphi \text{ implies } \Phi \not\models \varphi$$
$$\iff \text{if } \Phi \cup \{\neg\varphi\} \text{ is consistent, then } \Phi \cup \{\neg\varphi\} \text{ is satisfiable.}$$

As a matter of fact, we actually will prove the following general statement.

Theorem 3.2. cons(Φ) *implies that Φ is satisfiable.* $\dashv$

3.1. Henkin's Theorem.

We fix a set Φ of S-formulas and will construct an S-interpretation out of Φ. To that end, we first define a binary relation over the set T^S of S-terms.

Definition 3.3. Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. $\dashv$

Lemma 3.4. *(i) $\sim$ is an* equivalence *relation.*

(ii) $\sim$ is a congruence *relation. That is:*

- *For every n-ary function symbol $f \in S$ and $2 \cdot n$ S-terms $t_1 \sim t_1', \ldots, t_n \sim t_n'$, we have*

$$ft_1 \cdots t_n \sim ft_1' \cdots t_n'.$$

- *For every n-ary relation symbol $R \in S$ and $2 \cdot n$ S-terms $t_1 \sim t_1', \ldots, t_n \sim t_n'$, we have*

$$\Phi \vdash Rt_1 \cdots t_n \quad \iff \quad \Phi \vdash Rt_1' \cdots t_n'.$$

$\dashv$

Proof: By the equality rule and the substitution rule. □

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of t.

Definition 3.5. The *term structure for* Φ, denoted by $\mathfrak{T}^\Phi$, is defined as follows.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

(ii) For every n-ary relation symbol $R \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \ldots, \bar{t}_n) \in R^{\mathfrak{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \ldots t_n.$$

(iii) For every n-ary function symbol $f \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) := \overline{ft_1 \cdots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathfrak{T}^\Phi} := \bar{c}.$$

This finishes the construction of $\mathfrak{T}^\Phi$. $\dashv$

Using Lemma 3.4, in particular (ii), it is easy to verify that:

Lemma 3.6. $\mathfrak{T}^{\Phi}$ *is well-defined.* $\dashv$

To complete the definition of an S-interpretation, we still need to provide an assignment of the variables $v_0, v_1, \ldots$ in $\mathfrak{T}^{\Phi}$.

Definition 3.7. For every variable v_i we let

$$\beta^{\Phi}(v_i) := \bar{v}_i. \qquad \dashv$$

Finally we let

$$\mathfrak{I}^{\Phi} := \left(\mathfrak{T}^{\Phi}, \beta^{\Phi}\right).$$

Lemma 3.8. *(i) For any* $t \in T^S$

$$\mathfrak{I}^{\Phi}(t) = \bar{t}.$$

(ii) For every atomic φ

$$\mathfrak{I}^{\Phi} \models \varphi \quad \iff \quad \Phi \vdash \varphi.$$

Proof: (i) We proceed by induction on t.

- $t = v_i$ is a variable. Then
$$\mathfrak{I}^{\Phi}(v_i) = \beta^{\Phi}(v_i) = \bar{v}_i.$$
- $t = c$ is a constant. Then
$$\mathfrak{I}^{\Phi}(c) = c^{\mathfrak{T}^{\Phi}} = \bar{c}$$
- $t = ft_1 \cdots t_n$. Then
$$\begin{aligned} \mathfrak{I}^{\Phi}(ft_1 \cdots t_n) &= f^{\mathfrak{T}^{\Phi}}(\mathfrak{I}^{\Phi}(t_1), \ldots, \mathfrak{I}^{\Phi}(t_n)) \\ &= f^{\mathfrak{T}^{\Phi}}(\bar{t}_1, \ldots, \bar{t}_n) && \text{(by induction hypothesis)} \\ &= \overline{ft_1 \cdots t_n}. \end{aligned}$$

(ii) Recall that there are two types of atomic formulas. For the first type, let $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned} \mathfrak{I}^{\Phi} \models t_1 \equiv t_2 &\iff \mathfrak{I}^{\Phi}(t_1) = \mathfrak{I}^{\Phi}(t_2) \\ &\iff \bar{t}_1 = \bar{t}_2 && \text{(by (i))} \\ &\iff t_1 \sim t_2 \\ &\iff \Phi \vdash t_1 \equiv t_2. \end{aligned}$$

Second, let $\varphi = Rt_1 \cdots t_n$. We deduce

$$\begin{aligned} \mathfrak{I}^{\Phi} \models Rt_1 \cdots t_n &\iff \left(\mathfrak{I}^{\Phi}(t_1), \ldots, \mathfrak{I}^{\Phi}(t_n)\right) \in R^{\mathfrak{T}^{\Phi}} \\ &\iff \left(\bar{t}_1, \ldots, \bar{t}_n\right) \in R^{\mathfrak{T}^{\Phi}} && \text{(by (i))} \\ &\iff \Phi \vdash Rt_1 \cdots t_n. \end{aligned}$$

□

4. Exercises

Exercise 4.1. Can you derive the rule of contradiction from the modified contradiction?

Exercise 4.2. Prove:

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad \neg\neg\varphi} \qquad\qquad \text{(b)}\ \frac{\Gamma \quad \neg\neg\varphi}{\Gamma \quad \varphi}$$

Exercise 4.3. Is the following derivable?

$$\frac{}{\Gamma \quad \exists x\varphi \qquad \forall x\varphi}$$

Exercise 4.4. Let $S := \{R\}$ with unary relation symbol R. Moreover we define

$$\Phi := \{\exists x Rx\} \cup \{\neg Ry \mid \text{for every variable } y\}.$$

Prove that:

- Φ is consistent.
- There is no term $t \in T^S$ with $\Phi \vdash Rt$.

Exercise 4.5. Again let $S := \{R\}$ with unary relation symbol R, and

$$\Phi := \{Rx \vee Ry\}.$$

Prove that:

- Φ is consistent.
- $\Phi \nvdash Rx$ and $\Phi \nvdash Ry$.
- $\mathfrak{I}^\Phi \not\models \Phi$.