Mathematical Logic (VI)

Yijia Chen

1. Completeness

1.1. Henkin's Theorem. Recall that we fix a set Φ of S-formulas.

Definition 1.1. Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. $\dashv$

Lemma 1.2. *(i)* $\sim$ *is an* equivalence *relation.*

(ii) $\sim$ *is a* congruence *relation. That is:*

- *For every* n*-ary function symbol* $R \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$ft_1 \cdots t_n \sim ft_1' \cdots t_n'.$$

- *For every* n*-ary relation symbol* $R \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$\Phi \vdash Rt_1 \cdots t_n \iff \Phi \vdash Rt_1' \cdots t_n'.$$

$\dashv$

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of t.

Definition 1.3. The *term structure for* Φ, denoted by $\mathfrak{T}^\Phi$, is defined as below.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

(ii) For every n-ary relation symbol $R \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \ldots, \bar{t}_n) \in R^{\mathfrak{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \ldots t_n.$$

(iii) For every n-ary function symbol $f \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) := \overline{ft_1 \cdots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathfrak{T}^\Phi} := \bar{c}.$$

This finishes the construction of $\mathfrak{T}^\Phi$. For every variable v_i let

$$\beta^\Phi(v_i) := \bar{v}_i.$$

Finally we set

$$\mathfrak{I}^\Phi := \left(\mathfrak{T}^\Phi, \beta^\Phi\right),$$

which is the *term interpretation of* Φ. $\dashv$

Lemma 1.4. *(i) For any* $t \in T^S$

$$\mathfrak{I}^\Phi(t) = \bar{t}.$$

(ii) For every atomic φ

$$\mathfrak{I}^\Phi \models \varphi \iff \Phi \vdash \varphi. \qquad \dashv$$

Lemma 1.5. *Let* φ *be an S-formula and* $x_1, \ldots, x_n$ *pairwise distinct variables. Then*

(i) $\mathfrak{I}^\Phi \models \exists x_1 \ldots \exists x_n \varphi$ *if and only if there are S-terms* $t_1, \ldots, t_n$ *such that*

$$\mathfrak{I}^\Phi \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}.$$

(ii) $\mathfrak{I}^\Phi \models \forall x_1 \ldots \forall x_n \varphi$ *if and only if for all S-terms* $t_1, \ldots, t_n$ *we have*

$$\mathfrak{I}^\Phi \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}.$$

Proof: We prove (i), then (ii) follows immediately.

$$\begin{aligned}
\mathfrak{I}^\Phi \models & \exists x_1 \ldots \exists x_n \varphi \\
& \iff \mathfrak{I}^\Phi \frac{a_1 \ldots a_n}{x_1 \ldots x_n} \models \varphi \text{ for some } a_1, \ldots, a_n \in T^\Phi, \\
& \quad \text{i.e., } \mathfrak{I}^\Phi \frac{\bar{t}_1 \ldots \bar{t}_n}{x_1 \ldots x_n} \models \varphi \text{ for some } t_1, \ldots, t_n \in T^S, \\
& \iff \mathfrak{I}^\Phi \frac{\mathfrak{I}^\Phi(t_1) \ldots \mathfrak{I}^\Phi(t_n)}{x_1 \ldots x_n} \models \varphi \text{ for some } t_1, \ldots, t_n \in T^S, && \text{(by Lemma 1.4 (i))} \\
& \iff \mathfrak{I}^\Phi \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n} \text{ for some } t_1, \ldots, t_n \in T^S, && \text{(by the Substitution Lemma).}
\end{aligned}$$

□

Definition 1.6. (i) Φ is *negation complete* if for every S-formula φ

$$\Phi \vdash \varphi \quad \text{or} \quad \Phi \vdash \neg\varphi.$$

(ii) Φ contains witnesses if for every S-formula φ and every variable x there is a term $t \in T^S$ with

$$\Phi \vdash \left(\exists x \varphi \to \varphi \frac{t}{x} \right). \qquad \dashv$$

Lemma 1.7. *Assume that* Φ *is consistent, negation complete, and contains witnesses. Then for all S-formulas* φ *and* ψ*:*

(i) $\Phi \vdash \varphi$ *if and only if* $\Phi \nvdash \neg\varphi$.

(ii) $\Phi \vdash (\varphi \vee \psi)$ *if and only if* $\Phi \vdash \varphi$ *or* $\Phi \vdash \psi$.

(iii) $\Phi \vdash \exists x \varphi$ *if and only if there is a term* $t \in T^s$ *such that* $\Phi \vdash \varphi\frac{t}{x}$.

Proof: (i) Assume that $\Phi \vdash \varphi$. Since Φ is consistent, we conclude that $\Phi \nvdash \neg\varphi$. Conversely, if $\Phi \nvdash \neg\varphi$, then $\Phi \vdash \varphi$ by the negation completeness.

(ii) The direction from right to left is trivial by $\vee$-introduction in succedent. For the other direction, assume that $\Phi \vdash (\varphi \vee \psi)$ and $\Phi \nvdash \varphi$. By the negation completeness, $\Phi \vdash \neg\varphi$. Then for some finite $\Gamma \subseteq \Phi$ we have the following sequent proof.

	$\vdots$		
m.	Γ_1	$(\varphi \vee \psi)$	
	$\vdots$		
n.	Γ_2	$\neg\varphi$	
$(n+1)$.	$\Gamma_1\ \Gamma_2\ \varphi$	$\neg\varphi$	(antecedent by m)
$(n+2)$.	$\Gamma_1\ \Gamma_2\ \varphi$	φ	(assumption)
$(n+3)$.	$\Gamma_1\ \Gamma_2\ \varphi$	ψ	(modified contradiction by $n+1$ and $n+2$)
$(n+4)$.	$\Gamma_1\ \Gamma_2\ \psi$	ψ	(assumption)
$(n+5)$.	$\Gamma_1\ \Gamma_2\ (\varphi \vee \psi)$	ψ	($\vee$-introduction in antecedent)
$(n+6)$.	$\Gamma_1\ \Gamma_2$	ψ	(chain rule by m and $n+5$)

(iii) Let $\Phi \vdash \exists x\varphi$ and Φ contain witnesses. Thus there is a term $t \in T^S$ such that

$$\Phi \vdash \left(\exists x\varphi \to \varphi\frac{t}{x}\right).$$

By Modus ponens, we conclude $\Phi \vdash \varphi\frac{t}{x}$. The converse is by the rule of the $\exists$-introduction in succedent. $\square$

Theorem 1.8 (Henkin's Theorem). *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then for every S-formula φ*

$$\mathfrak{I}^\Phi \models \varphi \quad\Longleftrightarrow\quad \Phi \vdash \varphi.$$

Proof: We proceed by induction on φ.

- φ is atomic. This is Lemma 1.4 (ii).
- $\varphi = \neg\psi$. Then

$$\begin{aligned}\mathfrak{I}^\Phi \models \neg\psi &\iff \mathfrak{I}^\Phi \not\models \psi \\ &\iff \Phi \nvdash \psi && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash \neg\psi && \text{(by Lemma 1.7 (i)).}\end{aligned}$$

- $\varphi = (\psi_1 \vee \psi_2)$. We deduce

$$\begin{aligned}\mathfrak{I}^\Phi \models (\psi_1 \vee \psi_2) &\iff \mathfrak{I}^\Phi \models \psi_1 \text{ or } \mathfrak{I}^\Phi \models \psi_2 \\ &\iff \Phi \vdash \psi_1 \text{ or } \Phi \vdash \psi_2 && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash (\psi_1 \vee \psi_2) && \text{(by Lemma 1.7 (ii)).}\end{aligned}$$

- $\varphi = \exists x\psi$.

$$\begin{aligned}\mathfrak{I}^\Phi \models \exists x\psi &\iff \mathfrak{I}^\Phi \models \psi\frac{t}{x} \text{ for some } t \in T^S && \text{(by Lemma 1.5)} \\ &\iff \Phi \vdash \psi\frac{t}{x} \text{ for some } t \in T^S && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash \exists x\psi && \text{(by Lemma 1.7 (iii)).}\end{aligned}$$

Here, note that the length of $\psi\frac{t}{x}$ could be well larger than that $\exists x\psi$. Thus, our induction is on the so-called *connectivity rank* of ψ, denoted by $\mathrm{rk}(\varphi)$, which is defined as follows:

$$\mathrm{rk}(\varphi) := \begin{cases} 0 & \text{if } \varphi \text{ is atomic,} \\ 1 + \mathrm{rk}(\psi) & \text{if } \varphi = \neg\psi, \\ 1 + \mathrm{rk}(\psi_1) + \mathrm{rk}(\psi_2) & \text{if } \varphi = (\psi_1 \vee \psi_2), \\ 1 + \mathrm{rk}(\psi) & \text{if } \varphi = \exists x\psi. \end{cases}$$

$\square$

Corollary 1.9. *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then*

$$\mathfrak{I}^\Phi \models \Phi.$$

In particular, Φ is satisfiable.

1.2. The countable case. We fix a symbol set S which is at most countable. As a consequence, both T^S and L^S are countable. Let $\Phi \subseteq L^S$ we define

$$\text{free}(\Phi) := \bigcup_{\varphi \in \Phi} \text{free}(\varphi).$$

We will prove the following two lemmas.

Lemma 1.10. *Let $\Phi \subseteq L^S$ be consistent with* finite $\text{free}(\Phi)$. *Then there is a consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ such that Ψ contains witnesses.*

Lemma 1.11. *Let $\Psi \subseteq L^S$ be consistent. Then there is a consistent Θ with $\Psi \subseteq \Theta \subseteq L^S$ such that Θ is negation complete.*

Corollary 1.12. *Let $\Phi \subseteq L^S$ be consistent with finite* $\text{free}(\Phi)$. *Then Φ is satisfiable.*

Proof: By Lemma 1.10, there is a consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ such that Ψ contains witnesses. Observes that any Θ with $\Psi \subseteq \Theta \subseteq L^S$ still contains witnesses. By Lemma 1.11 we can choose such a Θ which is negation complete. Now Corollary 1.9 implies that Θ is satisfiable. Since $\Phi \subseteq \Theta$, Φ is satisfiable as well. $\square$

Proof of Lemma 1.10: Recall L^S is countable, thus we can enumerate all S-formulas

$$\exists x_0 \varphi_0, \exists x_1 \varphi_1, \ldots,$$

which start with an existential quantifier. Then we define inductively for every $n \in \mathbb{N}$ an S-formula ψ_n as follows. Assume that ψ_m has been defined for all $m < n$. Let

$$i_n := \min\big\{i \in \mathbb{N} \;\big|\; v_i \notin \text{free}\big(\Phi \cup \{\psi_m \mid m < n\} \cup \{\exists x_n \varphi_n\}\big)\big\}.$$

That is, i_n is the smallest index i such that v_i is not free in $\Phi \cup \{\psi_m \mid m < n\} \cup \{\exists x_n \varphi_n\}$. Then we set

$$\psi_n := \left(\exists x_n \varphi_n \to \varphi_n \frac{v_{i_n}}{x_n}\right).$$

Next, let

$$\Phi_n := \Phi \cup \big\{\psi_m \;\big|\; m < n\big\},$$

and $\Phi := \bigcup_{n \in \mathbb{N}} \Phi_n$. It should be clear that Φ contains witness. So what remains is to show that Φ is consistent, or equivalently every Φ_n is consistent.

Recall that $\Phi_0 = \Phi$ is consistent by our assumption. Towards a contradiction, assume that Φ_n is consistent, but Φ_{n+1} is not. Therefore, for every χ with $v_{i_n} \notin \text{free}(\exists x_n \varphi_n)$ there is a finite $\Gamma \subseteq \Phi_n$ with the following deduction.

$$\vdots$$

$m.$	$\Gamma \quad \left(\neg\exists x_n\varphi_n \vee \varphi_n\frac{v_{i_n}}{x_n}\right)$	χ	
$(m+1).$	$\Gamma \quad \neg\exists x_n\varphi_n$	$\neg\exists x_n\varphi_n$	(assumption)
$(m+2).$	$\Gamma \quad \neg\exists x_n\varphi_n$	$\left(\neg\exists x_n\varphi_n \vee \varphi_n\frac{v_{i_n}}{x_n}\right)$	($\vee$-introduction in the succedent)
$(m+3).$	$\Gamma \quad \neg\exists x_n\varphi_n$	χ	(chain rule)
$(m+4).$	$\Gamma \quad \varphi_n\frac{v_{i_n}}{x_n}$	χ	(similarly)
$(n+5).$	$\Gamma \quad \exists x_n\varphi_n$	χ	($\exists$-introduction in the antecedent)
$(m+6).$	Γ	χ	(case analysis).

Now by taking $\chi := \exists v_0 v_0 \equiv v_0$ and $\chi := \neg\exists v_0 v_0 \equiv v_0$ we conclude that Φ_n is inconsistent, which contradicts our assumption. □

Proof of Lemma 1.11: Let $\varphi_0, \varphi_1, \dots$ be an enumeration of L^S. For every $n \in \mathbb{N}$ we define Θ_n by induction. First $\Theta_0 := \Psi$. Then,

$$\Theta_{n+1} := \begin{cases} \Theta_n \cup \{\varphi_n\} & \text{if } \Theta_n \cup \{\varphi_n\} \text{ is consistent,} \\ \Theta_n & \text{otherwise.} \end{cases}$$

It is immediate that every Θ_n is consistent, and the consistency of

$$\Theta := \bigcup_{n\in\mathbb{N}} \Theta_n$$

follows. To see that Θ is negation complete, let $\varphi \in L^S$, in particular $\varphi = \varphi_n$ for some $n \in \mathbb{N}$. Assuming $\Theta \nvdash \neg\varphi_n$, we conclude $\Theta_n \nvdash \neg\varphi_n$ by $\Theta_n \subseteq \Theta$. Therefore, $\Theta_n \cup \{\varphi\}$ is consistent. It follows that $\varphi \in \Theta_{n+1} \subseteq \Theta$, and thus $\Theta \vdash \varphi$. □

2. Exercises

The exercise below shows that the finiteness of free(Φ) is necessary in Lemma 1.10.

Exercise 2.1. Let

$$\Phi := \left\{v_0 \equiv t \;\middle|\; t \in T^S\right\} \cup \left\{\exists v_0 \exists v_1 \neg v_0 \equiv v_1\right\}.$$

Prove that Φ is consistent, but there is no consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ which contains witnesses. ⊣

Exercise 2.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two S-structures. A mapping $h : A \to B$ is a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ if the following properties hold.

(1) For every n-ary relation symbol $R \in S$ and $a_1, \dots, a_n \in A$ we have

$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \text{ implies } (h(a_1), \dots, h(a_n)) \in R^{\mathfrak{B}}.$$

(2) For every n-ary function symbol $R \in S$ and $a_1, \dots, a_n \in A$ we have

$$h(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)).$$

(3) For every constant $c \in S$

$$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

Now let $\Phi \subseteq L^S$ and $\mathfrak{A}$ be an S-structure with $\mathfrak{A} \models \Phi$. Prove that there is a homomorphism from the term model $\mathfrak{T}^{\Phi}$ to $\mathfrak{A}$. ⊣