Mathematical Logic (VII)

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1. Completeness

Recall that we have shown:

Lemma 1.1. *Let* $\Phi \subseteq L^S$ *and* $\mathfrak{I}^\Phi$ *be the term interpretation of* Φ*. Then for every* atomic φ

$$\mathfrak{I}^\Phi \models \varphi \iff \Phi \vdash \varphi. \qquad \dashv$$

Theorem 1.2 (Henkin's Theorem)**.** *Let* $\Phi \subseteq L^S$ *be consistent, negation complete, and contain witnesses. Then for every S-formula* φ

$$\mathfrak{I}^\Phi \models \varphi \iff \Phi \vdash \varphi. \qquad \dashv$$

Corollary 1.3. *Let S be countable and* $\Phi \subseteq L^S$ *consistent with finite* $\mathrm{free}(\Phi)$*. Then there is a* Θ *such that*

- $\Phi \subseteq \Theta \subseteq L^S$*;*
- Θ *is consistent, negation complete, and contains witnesses.*

Therefore by Theorem 1.2 for every $\varphi \in L^S$

$$\mathfrak{I}^\Theta \models \varphi \iff \Theta \vdash \varphi.$$

In particular

$$\mathfrak{I}^\Theta \models \Phi,$$

thus Φ *is satisfiable.* $\dashv$

In the next step we eliminate the condition $\mathrm{free}(\Phi)$ being finite.

Corollary 1.4. *Let S be countable and* $\Phi \subseteq L^S$ *consistent. Then* Φ *is satisfiable.*

Proof: First, we let

$$S' := S \cup \{c_0, c_1, \ldots\}.$$

For every $\varphi \in L^S$ we define

$$n(\varphi) := \min\{n \mid \mathrm{free}(\varphi) \subseteq \{v_0, \ldots, v_{n-1}\}, \text{ i.e., } \varphi \in L^S_n\},$$

and let

$$\varphi' := \varphi \frac{c_0 \ldots c_{n(\varphi)-1}}{v_0 \ldots v_{n(\varphi)-1}}.$$

Then we set

$$\Phi' := \{\varphi' \mid \varphi \in \Phi\} \subseteq L^{S'}$$

Note $\mathtt{free}(\Phi') = \emptyset$.

Claim. Φ' is consistent.

Once we establish the claim, together with $\text{free}(\Phi') = \emptyset$, Corollary 1.3 implies that there is an S'-interpretation $\mathfrak{I}' = (\mathfrak{A}', \beta')$ such that $\mathfrak{I}' \models \Phi'$. Applying the Coincidence Lemma with $\text{free}(\Phi') = \emptyset$, we can assume without loss of generality that

$$\beta'(v_i) = c_i^{\mathfrak{A}'} = \mathfrak{I}'(c_i). \tag{1}$$

It follows that for every $\varphi \in \Phi$

$$\begin{aligned}
\mathfrak{I}' \models \varphi' &\iff \mathfrak{I}' \models \varphi\frac{c_0 \dots c_{n(\varphi)-1}}{v_0 \dots v_{n(\varphi)-1}} \\
&\iff \mathfrak{I}'\frac{\mathfrak{I}'(c_0) \dots \mathfrak{I}'(c_{n(\varphi)-1})}{v_0 \dots v_{n(\varphi)-1}} \models \varphi && \text{(by the Substitution Lemma)} \\
&\iff \mathfrak{I}'\frac{\beta'(v_0) \dots \beta'(v_{n(\varphi)-1})}{v_0 \dots v_{n(\varphi)-1}} \models \varphi && \text{(by (1))} \\
&\quad \text{i.e., } \mathfrak{I}' \models \varphi.
\end{aligned}$$

We conclude that Φ is satisfiable.

Now we prove the claim. It suffices to show that every finite subset of Φ' is satisfiable. To that end, let

$$\Phi'_0 := \{\varphi'_1, \dots, \varphi'_n\},$$

where $\varphi_1, \dots, \varphi_n \in \Phi$. Clearly $\text{free}\big(\{\varphi_1, \dots, \varphi_n\}\big)$ is finite, and $\{\varphi_1, \dots, \varphi_n\}$ is consistent by the consistency of Φ. By Corollary 1.3 there is an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ such that for every $i \in [n]$

$$\mathfrak{I} \models \varphi_i. \tag{2}$$

We expand the S-structure $\mathfrak{A}$ to an S'-structure $\mathfrak{A}'$ by setting for every $i \in \mathbb{N}$

$$c_i^{\mathfrak{A}'} := \beta(v_i). \tag{3}$$

Then for the S'-interpretation $\mathfrak{I}' := (\mathfrak{A}', \beta)$ and any $\varphi \in L^S$

$$\begin{aligned}
\mathfrak{I}' \models \varphi' &\iff \mathfrak{I}' \models \varphi\frac{c_0 \dots v_{n(\varphi)-1}}{v_0 \dots v_{n(\varphi)-1}} \\
&\iff \mathfrak{I}'\frac{\mathfrak{I}'(c_0) \dots \mathfrak{I}'(v_{n(\varphi)-1})}{v_0 \dots v_{n(\varphi)-1}} \models \varphi && \text{(by the Substitution Lemma)} \\
&\iff \mathfrak{I}'\frac{c_0^{\mathfrak{A}'} \dots v_{n(\varphi)-1}^{\mathfrak{A}'}}{v_0 \dots v_{n(\varphi)-1}} \models \varphi \\
&\iff \mathfrak{I}'\frac{\beta(v_0) \dots \beta(v_{n(\varphi)-1})}{v_0 \dots v_{n(\varphi)-1}} \models \varphi && \text{(by (3))} \\
&\iff \mathfrak{I}' \models \varphi \\
&\iff \mathfrak{I} \models \varphi && \text{(by the Coincidence Lemma).}
\end{aligned}$$

It follows that $\mathfrak{I}' \models \Phi'_0$ by (2). Thus Φ'_0 is satisfiable. □

1.1. The general case.

Lemma 1.5. *Let $\Phi \subseteq L^S$ be consistent. Then there is a symbol set S' with $S \subseteq S'$ and a consistent Ψ with $\Phi \subseteq \Psi \subseteq L^{S'}$ such that Ψ contains witnesses.* ⊣

Lemma 1.6. *Let $\Psi \subseteq L^S$ be consistent. Then there is a consistent Θ with $\Psi \subseteq \Theta \subseteq L^S$ such that Θ is negation complete.* ⊣

Then the next corollary follows from Lemmas 1.5 and 1.6 in the same fashion as that of Corollary 1.3.

Corollary 1.7. *Let $\Phi \subseteq L^S$ be consistent. Then Φ is satisfiable.* $\dashv$

We need some technical tools for proving Lemma 1.5. Let S be an arbitrary symbol set. For every $\varphi \in L^S$ we introduce a new constant $c_\varphi \notin S$. In particular, $c_\varphi \neq c_\psi$ for any $\varphi \neq \psi$. Then we set

$$S^* := S \cup \left\{ c_{\exists x \varphi} \mid \exists x \varphi \in L^S \right\},$$

$$W(S) := \left\{ \exists x \varphi \to \varphi \frac{c_{\exists x \varphi}}{x} \;\middle|\; \exists x \varphi \in L^S \right\}.$$

It is obvious that $c_{\exists x \varphi}$ is introduced as a witness for $\exists x \varphi$ as required by $W(S)$. Nevertheless, we pay a price for expanding the symbol set S to S^*, i.e., there are formulas of the form $\exists x \varphi$ in $L^{S^*} \setminus L^S$, e.g.,

$$\exists v_7 c_{\exists x Rx} \equiv v_7.$$

Lemma 1.8. *Assume that $\Phi \subseteq L^S$ is consistent. Then*

$$\Phi \cup W(S) \subseteq L^{S^*}$$

is consistent as well.

Proof: It suffices to show that every finite subset Φ_0^* of $\Phi \cup W(S) \subseteq L^{S^*}$ is satisfiable. Let

$$\Phi_0^* = \Phi_0 \cup \left\{ \exists x_1 \varphi_1 \to \varphi_1 \frac{c_1}{x_1}, \ldots, \exists x_n \varphi_n \to \varphi_n \frac{c_n}{x_n} \right\},$$

where $\Phi_0 \subseteq \Phi$ is finite, every $\exists x_i \varphi_i \in L^S$, and $c_i = c_{\exists x_i \varphi_i}$ for $i \in [n]$.

Choose a finite $S_0 \subseteq S$ such that $\Phi_0 \subseteq L^{S_0}$. Note that Φ_0 is consistent due to the consistency of Φ. Furthermore free(Φ_0) is finite[1]. Therefore Φ is satisfiable by Corollary 1.3, i.e., there is an S_0-interpretation $\mathfrak{I}_0 = (\mathfrak{A}_0, \beta)$ such that

$$\mathfrak{I}_0 \models \Phi_0$$

Note that $\mathfrak{A}_0$ is an S_0-structure. By choosing some arbitrary interpretation of the symbols in $S \setminus S_0$ we obtain an S-structure $\mathfrak{A}$. Then the Coincidence Lemma guarantees that for the S-interpretation $\mathfrak{I} := (\mathfrak{A}, \beta)$

$$\mathfrak{I} \models \Phi_0.$$

Next, we need to further expand $\mathfrak{A}$ to an S^*-structure $\mathfrak{A}^*$ by giving interpretation of all new constants $c_{\exists x \varphi}$. Let $a \in A$ be an arbitrary but fixed element. Then for every $i \in [n]$ we set

$$c_i^{\mathfrak{A}^*} := \begin{cases} a_i & \text{if there is an } a_i \in A \text{ with } \mathfrak{I} \models \varphi_i \frac{a_i}{x_i}, \\ & \quad \text{(choose an arbitrary one, if there are more than one such } a_i\text{)}, \\ a & \text{otherwise.} \end{cases}$$

For all the other new constants $c_{\exists x \varphi}$ we simply let $c_{\exists x \varphi}^{\mathfrak{A}^*} := a$. Then for the S^*-interpretation $\mathfrak{I}^* := (\mathfrak{A}^*, \beta)$ we claim

$$\mathfrak{I}^* \models \Phi_0 \cup \left\{ \exists x_1 \varphi_1 \to \varphi_1 \frac{c_1}{x_1}, \ldots, \exists x_n \varphi_n \to \varphi_n \frac{c_n}{x_n} \right\}.$$

$\mathfrak{I}^* \models \Phi_0$ is immediate by $\mathfrak{I} \models \Phi_0$ and the Coincidence Lemma. Let $i \in [n]$ and assume $\mathfrak{I}^* \models \exists x_i \varphi_i$, or equivalently $\mathfrak{I} \models \exists x_i \varphi_i$. Then by our choice of $a_i \in A$

$$\mathfrak{I} \models \varphi_i \frac{a_i}{x_i},$$

[1]Here, we can also apply Corollary 1.4 without using the finiteness of free(Φ_0). But then this would introduce a further layer of construction as in the proof of Corollary 1.4.

hence

$$\mathfrak{I}^* \models \exists x_i \varphi_i \to \varphi_i \frac{c_i}{x_i}, \tag{4}$$

by the Coincidence Lemma and by the Substitution Lemma. Note (4) trivially holds if $\mathfrak{I}^* \not\models \exists x_i \varphi_i$. This finishes the proof. □

Lemma 1.9. *Let*

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots$$

be a sequence of symbol sets. Furthermore, for every $n \in \mathbb{N}$ *let* Φ_n *be a set of* S_n*-formulas such that*

$$\Phi_0 \subseteq \Phi_1 \subseteq \cdots \subseteq \Phi_n \subseteq \cdots$$

We set

$$S := \bigcup_{n \in \mathbb{N}} S_n \quad \textit{and} \quad \Phi := \bigcup_{n \in \mathbb{N}} \Phi_n.$$

Then Φ *is a consistent set of* S*-formulas if and only if every* Φ_n *is consistent.*

Proof: We prove that

$$\Phi \text{ is inconsistent} \iff \Phi_n \text{ is inconsistent for some } n \in \mathbb{N}.$$

The direction from right to left is trivial. So assume that Φ is inconsistent. In particular, for some $\varphi \in L^S$ there are proofs of φ and $\neg\varphi$ from Φ. Since proofs in sequent calculus are all finite, we can choose a finite $S' \subseteq S$ such that every formula used in the proofs of φ and $\neg\varphi$ is an S'-formulas. For the same reason, for a sufficiently large $n \in \mathbb{N}$ we have

(i) $S' \subseteq S_n$,

(ii) $\Phi_n \vdash \varphi$ and $\Phi_n \vdash \neg\varphi$.

Thus Φ_n is inconsistent. □

Remark 1.10. Note at this point we have not shown the following seemingly trivial result. Let S be an (infinite) set of symbols, a finite $\Phi \subseteq L^S$, and $\varphi \in L^S$ such that $\Phi \vdash \varphi$. Furthermore, let $S_0 \subseteq S$ be the set of symbols that occur in Φ and φ. Then there is a proof of sequence calculus for $\Phi \vdash \varphi$ such that every formula occurs in the proof is an S_0-formula, i.e., only uses symbols in S_0.

This is the reason in the proof of Lemma 1.9 we need to emphasize (i). ⊣

Proof of Lemma 1.5: Let

$$S_0 := S \quad \text{and} \quad S_{n+1} := (S_n)^*,$$
$$\Psi_0 := \Phi \quad \text{and} \quad \Psi_{n+1} := \Psi_n \cup W(S_n).$$

Therefore

$$S = S_0 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$$
$$\Phi = \Psi_0 \subseteq \cdots \subseteq \Psi_n \subseteq \Psi_{n+1} \subseteq \cdots$$

Then we set

$$S' := \bigcup_{n \in \mathbb{N}} S_n \quad \text{and} \quad \Psi := \bigcup_{n \in \mathbb{N}} \Psi_n.$$

By Lemma 1.8 and induction on n we conclude that every Ψ_n is consistent. Thus Lemma 1.9 implies that Φ is a consistent set of S'-formulas.

By our construction of $W(S_n)$, the set Φ trivially contains witnesses. $\square$

The proof of Lemma 1.6 relies on well-known Zorn's Lemma. Let M be a set and $\mathcal{U} \subseteq \mathscr{P}ow(M) = \{T \mid T \subseteq M\}$. We say that a *nonempty* subset $C \subseteq \mathcal{U}$ is a *chain* in $\mathcal{U}$ if for every $T_1, T_2 \in C$ either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$.

Lemma 1.11 (Zorn's Lemma)**.** *Assume that for every chain C in $\mathcal{U}$ we have*

$$\bigcup C := \{a \mid a \in T \textit{ for some } T \in C\} \in \mathcal{U}.$$

Then $\mathcal{U}$ has a maximal element T*, i.e., there is no $T' \in \mathcal{U}$ with $T \subsetneq T'$.* $\dashv$

Proof of Lemma 1.6 In order to apply Zorn's Lemma we let $M := L^S$ and

$$\mathcal{U} := \{\Theta \mid \Psi \subseteq \Theta \subseteq L^S \text{ and } \Theta \text{ is consistent}\}.$$

Let C be a chain in $\mathcal{U}$. We set

$$\Theta_C := \bigcup C = \{\varphi \mid \varphi \in \Theta \text{ for some } \Theta \in C\}.$$

$C \neq \emptyset$ implies $\Psi \subseteq \Theta_C$. To see that Θ_C is consistent, let $\{\varphi_1, \dots, \varphi_n\}$ be a finite subset of Θ_C, in particular, there are $\Theta_i \in C$ such that $\varphi_i \in \Theta_i$. As C is a chain, without loss of generality, we can assume that every $\Theta_i \subseteq \Theta_n$. Since $\Theta_n \in C$ is consistent by the definition of $\mathcal{U}$, we conclude $\{\varphi_1, \dots, \varphi_n\}$ is consistent as well.

Thus the condition in Zorn's Lemma is satisfied. It follows that $\mathcal{U}$ has a maximal element Θ. We claim that Θ is negation complete. Otherwise, for some $\varphi \in L^S$ we have $\Theta \nvdash \varphi$ and $\Theta \nvdash \neg\varphi$. Therefore $\varphi \notin \Theta$ and $\Theta \cup \{\varphi\}$ is consistent. As a consequence $\Theta \subsetneq \Theta \cup \{\varphi\} \in \mathcal{U}$. This is a contradiction to the maximality of Θ. $\square$

Now we are ready to prove the completeness theorem.

Theorem 1.12. *Let $\Phi \subseteq L^S$ and $\varphi \in L^S$. Then*

$$\Phi \vdash \varphi \quad \Longleftrightarrow \quad \Phi \models \varphi.$$

Proof: The direction from left to right is easy by the definition of sequent calculus. Conversely, assume that $\Phi \nvdash \varphi$, then $\Phi \cup \neg\{\neg\varphi\}$ is consistent. Corollary 1.7 implies that $\Phi \cup \neg\{\neg\varphi\}$ is satisfiable. In particular, there is an S-interpretation $\mathfrak{I}$ with $\mathfrak{I} \models \Phi$ and $\mathfrak{I} \models \neg\varphi$ (i.e., $\mathfrak{I} \not\models \varphi$). But this means that $\Phi \not\models \varphi$. $\square$

2. Exercises

Prove Remark 1.10, that is:

Exercise 2.1. Let $\Phi \subseteq L^S$ be finite, and let $\varphi \in L^S$ with $\Phi \vdash \varphi$. Note that a proof might use formulas built on any symbol in S.

Define $S_0 \subseteq S$ to be the set of symbols that occur in Φ and φ. Then there is a proof for $\Phi \vdash \varphi$ such that every formula occurs in the proof is an S_0-formula. $\dashv$

Definition 2.2. A *total order* on a set A is a binary relation $\leqslant \subseteq A \times A$ with the following properties. Let $a, b, c \in A$ be arbitrary.

(i) $a \leqslant a$ (i.e., $\leqslant$ is reflexive).

(ii) If $a \leqslant b$ and $b \leqslant a$, then $a = b$ (i.e., $\leqslant$ is anti-symmetric).

(iii) If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$ (i.e., $\leqslant$ is transitive).

(iv) $a \leqslant b$ or $b \leqslant a$ (i.e., $\leqslant$ is total).

If furthermore

(v) every nonempty $A' \subseteq A$ has a *minimum* element a, i.e., $a \in A'$ and $a \leqslant a'$ for any $a' \in A'$,

then $\leqslant$ is a *well order*. $\dashv$

Exercise 2.3. Assume that for every set A there is a well order $\leqslant \subseteq A \times A$. Prove Zorn's Lemma. $\dashv$