

Mathematical Logic (VIII)

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1. The Löwenheim-Skolem Theorem and the Compactness Theorem

Using the term-interpretation, it is routine to verify:

Theorem 1.1 (Löwenheim-Skolem). *Let $\Phi \subseteq L^S$ be at most countable and satisfiable. Then there is an S -interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ such that*

- *the universe A of \mathfrak{A} is at most countable,*
- *and $\mathfrak{I} \models \Phi$.* ⊢

The following is a more general version.

Theorem 1.2 (Downward Löwenheim-Skolem). *Let $\Phi \subseteq L^S$ be satisfiable. Then there is an S -interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ such that*

- *$|A| \leq |\mathfrak{T}^S| = |L^S|$,*
- *and $\mathfrak{I} \models \Phi$.* ⊢

Corollary 1.3. *Let $S := \{+, \times, <, 0, 1\}$ with the usual meaning and*

$$\Phi_{\mathbb{R}} := \{ \varphi \in L_0^S \mid (\mathbb{R}, +, \cdot, <, 0, 1) \models \varphi \}.$$

Then there is a countable S -structure \mathfrak{A} with $\mathfrak{A} \models \Phi_{\mathbb{R}}$. ⊢

By the Completeness Theorem:

Theorem 1.4 (Compactness). (a) $\Phi \models \varphi$ if and only if there is a finite $\Phi_0 \subseteq \Phi$ with $\Phi_0 \models \varphi$.

(b) Φ is satisfiable if and only if every finite $\Phi_0 \subseteq \Phi$ is satisfiable. ⊢

In fact, the “compactness” is a notion from topology. We can explain the topological perspective of Theorem 1.4 using *finite covers* from analysis. For every $\varphi \in L^S$ we define

$$\text{Mod}(\varphi) := \{ \mathfrak{I} \mid \mathfrak{I} \models \varphi \},$$

and

$$\text{Mod}(\Phi) := \{ \mathfrak{I} \mid \mathfrak{I} \models \Phi \} = \bigcap_{\psi \in \Phi} \text{Mod}(\psi).$$

We show that Theorem 1.4 is equivalent to the following *finite cover property*.

Proposition 1.5. $\text{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi} \text{Mod}(\psi)$ if and only if for some finite $\Phi_0 \subseteq \Phi$ we have

$$\text{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi_0} \text{Mod}(\psi). \quad \text{⊢}$$

Proof of Theorem 1.4 using Proposition 1.5:

$$\begin{aligned}
\Phi \models \varphi &\iff \text{Mod}(\Phi) \subseteq \text{Mod}(\varphi) \\
&\iff \overline{\text{Mod}(\varphi)} \subseteq \overline{\text{Mod}(\Phi)} \\
&\iff \overline{\text{Mod}(\varphi)} \subseteq \bigcap_{\psi \in \Phi} \overline{\text{Mod}(\psi)} \\
&\iff \overline{\text{Mod}(\varphi)} \subseteq \bigcup_{\psi \in \Phi} \overline{\text{Mod}(\psi)} \\
&\iff \text{Mod}(\neg\varphi) \subseteq \bigcup_{\psi \in \Phi} \text{Mod}(\neg\psi) \\
&\iff \text{Mod}(\neg\varphi) \subseteq \bigcup_{\psi \in \Phi_0} \text{Mod}(\neg\psi) \text{ for some finite } \Phi_0 \subseteq \Phi \quad (\text{by Proposition 1.5}) \\
&\iff \overline{\text{Mod}(\varphi)} \subseteq \bigcup_{\psi \in \Phi_0} \overline{\text{Mod}(\psi)} \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \overline{\text{Mod}(\varphi)} \subseteq \overline{\bigcap_{\psi \in \Phi_0} \text{Mod}(\psi)} \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \bigcap_{\psi \in \Phi_0} \text{Mod}(\psi) \subseteq \text{Mod}(\varphi) \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \text{Mod}(\Phi_0) \subseteq \text{Mod}(\varphi) \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \Phi_0 \models \varphi \text{ for some finite } \Phi_0 \subseteq \Phi. \quad \square
\end{aligned}$$

Proof of Proposition 1.5 by Theorem 1.4: The direction from right to left is trivial. So we assume that

$$\text{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi} \text{Mod}(\psi).$$

Claim. $\{\neg\psi \mid \psi \in \Phi\} \models \neg\varphi$.

Proof of the claim. Let \mathcal{J} be an interpretation with

$$\mathcal{J} \models \{\neg\psi \mid \psi \in \Phi\}.$$

That is, $\mathcal{J} \models \neg\psi$ for every $\psi \in \Phi$. We can deduce that

$$\begin{aligned}
\mathcal{J} \in \bigcap_{\psi \in \Phi} \text{Mod}(\neg\psi) &\iff \mathcal{J} \in \bigcap_{\psi \in \Phi} \overline{\text{Mod}(\psi)} \\
&\iff \mathcal{J} \in \overline{\bigcup_{\psi \in \Phi} \text{Mod}(\psi)} \\
&\iff \mathcal{J} \notin \bigcup_{\psi \in \Phi} \text{Mod}(\psi) \\
&\implies \mathcal{J} \notin \text{Mod}(\varphi) \quad \left(\text{by } \text{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi} \text{Mod}(\psi) \right) \\
&\iff \mathcal{J} \models \neg\varphi.
\end{aligned}$$

This finishes the proof of the claim. \dashv

Now we apply Theorem 1.4 to the above claim. In particular, there is a finite $\Phi_0 \subseteq \Phi$ such that

$$\{\neg\psi \mid \psi \in \Phi_0\} \models \neg\varphi$$

Then arguing similarly as above, we obtain

$$\text{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi_0} \text{Mod}(\psi). \quad \square$$

Theorem 1.6. *Let $\Phi \subseteq L^S$ such that for every $n \in \mathbb{N}$ there exists an S -interpretation $\mathcal{I}_n = (\mathfrak{A}_n, \beta_n)$ with $|A_n| \geq n$ and $\mathcal{I}_n \models \Phi$. Then there is an S -interpretation $\mathcal{I} = (\mathfrak{A}, \beta)$ with infinite A and $\mathcal{I} \models \Phi$.*

Proof: For every $n \geq 2$ we define a sentence

$$\varphi_{\geq n} := \exists v_0 \cdots \exists v_{n-1} \bigwedge_{0 \leq i < j < n} \neg v_i \equiv v_j.$$

Clearly for any structure \mathfrak{A} (regardless of the symbol set S)

$$\mathfrak{A} \models \varphi_{\geq n} \iff |A| \geq n.$$

Now consider

$$\Psi := \Phi \cup \{\varphi_{\geq n} \mid n \geq 2\}.$$

Of course every finite subset of Ψ is contained in

$$\Psi_{n_0} := \Phi \cup \{\varphi_{\geq n} \mid 2 \leq n \leq n_0\}$$

for a sufficiently large $n_0 \in \mathbb{N}$. By assumption, the interpretation \mathcal{I}_{n_0} witnesses that Ψ_{n_0} is satisfiable. Therefore, by the Compactness Theorem, Ψ itself is satisfiable. The result follows immediately. \square

Theorem 1.7 (Upward Löwenheim-Skolem). *Let $\Phi \subseteq L^S$ and assume that there is an S -interpretation $\mathcal{I} = (\mathfrak{A}, \beta)$ such that A is infinite and $\mathcal{I} \models \Phi$. Then, for any set B there is an S -interpretation $\mathcal{J} = (\mathfrak{A}, \beta)$ with $|A| \geq |B|$ and $\mathcal{J} \models \Phi$.*

Proof: For any $b \in B$ we introduce a new constant $c_b \notin S$. In particular, $c_b \neq c_{b'}$ for any $b, b' \in B$ with $b \neq b'$. Then consider

$$\Psi := \Phi \cup \{\neg c_b \equiv c_{b'} \mid b, b' \in B \text{ with } b \neq b'\}.$$

Since Φ has an infinite interpretation, every finite subset of Ψ is satisfiable. By the Compactness Theorem, we conclude that Ψ is satisfiable. Clearly the structure in any interpretation which satisfies Ψ must have size as large as $|B|$. \square

Corollary 1.8. *Let $S = \{+, \times, <, 0, 1\}$ and*

$$\Phi_{\mathbb{N}} := \{\varphi \in L_0^S \mid (\mathbb{N}, +, \cdot, <, 0, 1) \models \varphi\}.$$

Then there is a uncountable S -structure \mathfrak{A} with $\mathfrak{A} \models \Phi_{\mathbb{N}}$. \dashv

2. Decidability and Enumerability

A. Procedure and Decidability.

Definition 2.1. Let \mathcal{A} be an alphabet (which we always assume to be finite) and $W \subseteq \mathcal{A}^*$.

- (i) Let \mathbb{P} be a procedure/program (which we will make precise shortly afterwards). \mathbb{P} is a *decision procedure for W* if on every input $w \in \mathcal{A}^*$ the procedure \mathbb{P} will eventually halt and output some $w' \in \mathcal{A}^*$ such that

- if $w \in W$, then $w' = \square$, where \square is the empty string,
- if $w \notin W$, then $w' \neq \square$.

(ii) W is *decidable* if there is a decision procedure for W . ⊢

B. Enumerability.

Definition 2.2. Let \mathcal{A} be an alphabet and $W \subseteq \mathcal{A}^*$.

(i) A procedure \mathbb{P} is an *enumeration procedure for W* if \mathbb{P} (without any input) outputs all the words in W (in some order and possibly with repetitions).

(ii) W is *enumerable* if there is an enumeration procedure for W . ⊢

Lemma 2.3. *If there is an enumeration procedure for W , then there is an enumeration procedure for W without repetitions.* ⊢

Lemma 2.4. *Let \mathcal{A} be finite. Then \mathcal{A}^* is enumerable.* ⊢

Let

$$\begin{aligned}
 S_\infty &:= \{c_0, c_1, \dots\} && \text{(every } c_i \text{ is a constant)} \\
 &\cup \bigcup_{n \geq 1} \{R_0^n, R_1^n, \dots\} && \text{(every } R_i^n \text{ is an } n\text{-ary relation symbol)} \\
 &\cup \bigcup_{n \geq 1} \{f_0^n, f_1^n, \dots\} && \text{(every } f_i^n \text{ is an } n\text{-ary function symbol).}
 \end{aligned}$$

Lemma 2.5.

$$\{\varphi \in L_0^{S_\infty} \mid \models \varphi\}$$

is *enumerable*.

Proof: [sketch] By the Completeness Theorem

$$\{\varphi \in L_0^{S_\infty} \mid \models \varphi\} = \{\varphi \in L_0^{S_\infty} \mid \vdash \varphi\}.$$

Thus, we can enumerate all possible proofs/derivations of symbol set S_∞ , thus obtain all those $\varphi \in L_0^{S_\infty}$ with $\vdash \varphi$. □

C. The Relationship between Decidability and Enumerability.

Theorem 2.6. *Every decidable set is enumerable.*

Proof: Assume that the procedure \mathbb{P} decides $W \subseteq \mathcal{A}^*$. By Lemma 2.4 we can enumerate all $w \in \mathcal{A}^*$. For each w we can decide whether $w \in W$ by calling \mathbb{P} . If so, we output w and proceed to the next string. Otherwise, we move to the next string without outputting w . □

We will see later that the converse of Theorem 2.6 does not hold, i.e., there are enumerable sets which are not decidable. Nevertheless, we can show:

Theorem 2.7. *Let $W \subseteq \mathcal{A}^*$. Then W is decidable if and only if both W and $\mathcal{A}^* \setminus W$ are enumerable.*

Proof: The direction from left to right is straightforward by Theorem 2.6 and by observing that $\mathcal{A}^* \setminus W$ is decidable as well. For the converse, we have two procedures, \mathbb{P}_1 which enumerates W , and \mathbb{P}_2 which enumerates $\mathcal{A}^* \setminus W$.

Then given an input $w \in \mathcal{A}^*$, we simulate two procedures \mathbb{P}_1 and \mathbb{P}_2 simultaneously¹, eventually w will appear in exactly one of the outputs of \mathbb{P}_1 and \mathbb{P}_2 . Then we can answer whether $w \in W$ accordingly. \square

D. Computable Functions.

Definition 2.8. Let \mathcal{A} and \mathcal{B} be two alphabets. A procedure that for each input $w \in \mathcal{A}^*$ outputs a $w' \in \mathcal{B}^*$ determines a function $f : \mathcal{A}^* \rightarrow \mathcal{B}^*$ defined by

$$w \xrightarrow{f} w'.$$

f is said to be *computable*. \dashv

3. Exercises

Not this time.

¹More precisely, we simulate the steps of \mathbb{P}_1 and \mathbb{P}_2 alternatively, i.e., the first step of \mathbb{P}_1 , the first step of \mathbb{P}_2 , the second step of \mathbb{P}_1 , the second step of \mathbb{P}_2 , ...