Peano Arithmetic

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1. Peano Arithmetic in First-Order Logic

We fix $S_{ar} = \{+, \cdot, 0, 1\}$ as our symbol set.

Definition 1.1. The *Peano Arithmetic* Φ_{PA} consists of the following sentences,

$$\begin{array}{ll} \forall x \neg x + 1 \equiv 0, & \forall x \forall y (x + 1 \equiv y + 1 \to x \equiv y), \\ \forall x\ x + 0 \equiv x, & \forall x \forall y\ x + (y + 1) \equiv (x + y) + 1, \\ \forall x\ x \cdot 0 \equiv 0, & \forall x \forall y\ x \cdot (y + 1) \equiv x \cdot y + x, \end{array}$$

and for all $n \in \mathbb{N}$, all variables $x_1, \ldots, x_n$, y, and all $\varphi \in L^{S_{ar}}$ with

$$\text{free}(\varphi) \subseteq \{x_1, \ldots, x_n, y\}$$

the sentence

$$\forall x_1 \cdots \forall x_n \left(\left(\varphi \frac{0}{y} \wedge \forall y \left(\varphi \to \varphi \frac{y+1}{y} \right) \right) \to \forall y \varphi \right). \qquad \dashv$$

Recall for every $n \geqslant 1$ we define

$$\bar{n} := \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}.$$

Also 0 is identified with $\bar{0}$.

Let t_1, t_2 be two terms. We use $t_1 \leqslant t_2$ to denote the formula

$$\exists x (t_1 + x \equiv t_2).$$

We will need the following tools, which are not hard to show.

Lemma 1.2. *(i) Let* t_1, t_2 *be two* closed *terms, i.e.,* $\text{var}(t_1) = \text{var}(t_2) = \emptyset$. *Then*

$$\mathfrak{N} \models t_1 \equiv t_2 \iff \Phi_{PA} \vdash t_1 \equiv t_2,$$

and

$$\mathfrak{N} \models \neg t_1 \equiv t_2 \iff \Phi_{PA} \vdash \neg t_1 \equiv t_2,$$

(ii) Let $n \in \mathbb{N}$. *Then*

$$\Phi_{PA} \vdash \forall x (x \leqslant \bar{n} \to x \equiv 0 \vee x \equiv 1 \vee \cdots \vee x \equiv \bar{n}).$$

(iii)

$$\Phi_{PA} \vdash \forall x \forall y (x \leqslant y \vee y \leqslant x).$$

2. Some Restricted First-order Formulas

2.1. Δ_0-formulas and Σ_1-formulas. For a variable x, a term t with $x \notin \mathrm{var}(t)$, and a formula ψ,

$$\exists x \leqslant t\ \psi$$

is the abbreviation for

$$\exists x \big(x \leqslant t \wedge \psi\big).$$

Similarly

$$\forall x \leqslant t\ \psi$$

denotes

$$\forall x \big(x \leqslant t \to \psi\big).$$

Definition 2.1 (Δ_0-formula). (i) For all terms t_1 and t_2, the formula $t_1 \equiv t_2$ is in Δ_0.

(ii) If $\psi \in \Delta_0$, then $\neg\psi \in \Delta_0$ as well.

(iii) If $\psi_1, \psi_2 \in \Delta_0$, then both $\psi_1 \vee \psi_2$ and $\psi_1 \wedge \psi_2$ are in Δ_0.

(iv) If $\psi \in \Delta_0$, then, for all variables x and y with $x \neq y$, both $\exists x \leqslant y\ \psi$ and $\forall x \leqslant y\ \psi$ are in Δ_0. ⊣

Definition 2.2 (Σ_1-formula). (i) Every Δ_0-formula is in Σ_1.

(ii) If $\psi \in \Sigma_1$, then, for every variable x the formula $\exists x\ \psi$ is in Δ_0. ⊣

Lemma 2.3. *Every Σ_1-formula is equivalent to a Σ_1-formula in which every negation only appears at the atomic level.* ⊣

2.2. The provability of Σ_1-formulas in Φ_{PA}.

Theorem 2.4. *Let $\varphi(x_1, \ldots, x_k) \in \Sigma_1$ with $\mathrm{free}(\varphi) \subseteq \{x_1, \ldots, x_k\}$. Then for any $n_1, \ldots, n_k \in \mathbb{N}$*

$$\mathfrak{N} \models \varphi(\bar{n}_1, \ldots, \bar{n}_k) \quad \Longleftrightarrow \quad \Phi_{\mathrm{PA}} \vdash \varphi(\bar{n}_1, \ldots, \bar{n}_k).$$

Proof: The direction from right to left follows from $\mathfrak{N} \models \Phi_{\mathrm{PA}}$. For the other direction, let $n_1, \ldots, n_k \in \mathbb{N}$ satisfy

$$\mathfrak{N} \models \varphi(\bar{n}_1, \ldots, \bar{n}_k). \tag{1}$$

We proceed by the induction on $\varphi(x_1, \ldots, x_k)$. By Lemma 2.3 we can assume that all negations in φ are at the atomic level.

- The case for atomic or negated atomic φ is precisely Lemma 1.2 (i).
- $\varphi = \psi_1 \vee \psi_2$. Then by (1)

  $$\mathfrak{N} \models \psi_1(\bar{n}_1, \ldots, \bar{n}_k) \quad \text{or} \quad \mathfrak{N} \models \psi_2(\bar{n}_1, \ldots, \bar{n}_k).$$

  So by induction hypothesis

  $$\Phi_{\mathrm{PA}} \vdash \psi_1(\bar{n}_1, \ldots, \bar{n}_k) \quad \text{or} \quad \Phi_{\mathrm{PA}} \vdash \psi_2(\bar{n}_1, \ldots, \bar{n}_k).$$

  In either case,

  $$\Phi_{\mathrm{PA}} \vdash \psi_1(\bar{n}_1, \ldots, \bar{n}_k) \vee \psi_2(\bar{n}_1, \ldots, \bar{n}),$$

  i.e.,

  $$\Phi_{\mathrm{PA}} \vdash \varphi(\bar{n}_1, \ldots, \bar{n}_k).$$

- $\varphi = \psi_1 \wedge \psi_2$ can be argued similarly as $\psi_1 \vee \psi_2$.
- $\varphi = \exists x \leqslant x_i\ \psi(x, x_1, \dots, x_k)$. By (1) there is an $n_0 \in \mathbb{N}$ with $n_0 \leqslant n_i$ such that

$$\mathfrak{N} \models \psi(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_k).$$

By induction hypothesis

$$\Phi_{\text{PA}} \vdash \psi(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_k).$$

Then Lemma 1.2 (i) implies

$$\Phi_{\text{PA}} \vdash \bar{n}_0 + \overline{n_i - n_0} \equiv \bar{n}_i.$$

Hence

$$\Phi_{\text{PA}} \vdash \bar{n}_0 \leqslant \bar{n}_i.$$

It follows that

$$\Phi_{\text{PA}} \vdash \exists x\big(x \leqslant \bar{n}_i \wedge \psi(x, \bar{n}_1, \dots, \bar{n}_k)\big), \qquad \text{i.e.,} \quad \Phi_{\text{PA}} \vdash \varphi(\bar{n}_1, \dots, \bar{n}_k).$$

- $\varphi = \forall x \leqslant x_i\ \psi(x, x_1, \dots, x_k)$. Then, (1) implies that for all $n_0 \leqslant n_i$ we have

$$\mathfrak{N} \models \psi(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_k).$$

By induction hypothesis, for all $n_0 \leqslant n_i$,

$$\Phi_{\text{PA}} \vdash \psi(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_k)$$

Recall Lemma 1.2 (ii)

$$\Phi_{\text{PA}} \vdash \forall x(x \leqslant \bar{n}_i \to x \equiv 0 \vee x \equiv 1 \vee \cdots \vee x \equiv \bar{n}_i).$$

Thus

$$\Phi_{\text{PA}} \vdash \forall x \leqslant \bar{n}_i\ \psi(x, \bar{n}_1, \dots, \bar{n}_k), \qquad \text{i.e.,} \quad \Phi_{\text{PA}} \vdash \varphi(\bar{n}_1, \dots, \bar{n}_k).$$

- $\varphi = \exists x \psi(x, x_1, \dots, x_k)$. By (1) there is an $n_0 \in \mathbb{N}$ such that

$$\mathfrak{N} \models \psi(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_k).$$

Again by induction hypothesis, we conclude

$$\Phi_{\text{PA}} \vdash \psi(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_k).$$

It follows that

$$\Phi_{\text{PA}} \vdash \exists x \psi(x, \bar{n}_1, \dots, \bar{n}_k).$$

□

2.3. Functional Δ_0-formulas.

Definition 2.5. A Δ_0-formula $\delta(x_1, \dots, x_r, x, y)$ is *functional* if for every $n_1, \dots, n_r \in \mathbb{N}$ there is a *unique* $n \in \mathbb{N}$ such that

$$\mathfrak{N} \models \exists y\big(\bar{n} \leqslant y \wedge \delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, y)\big).$$

Equivalently, we write

$$\theta_\delta(x_1, \dots, x_r, x, y) := x \leqslant y \wedge \delta, \tag{2}$$

and then

$$\mathfrak{N} \models \exists^{=1} x \exists y\ \theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x, y).$$

⊣

Definition 2.6. Let $\delta(x_1, \ldots, x_r, x_{r+1}, y) \in \Delta_0$ be functional. Then the function $f_\delta : \mathbb{N}^r \to \mathbb{N}$ is defined by

$$f_\delta(n_1, \ldots, n_r) := n,$$

where $n \in \mathbb{N}$ is the unique natural number such that $\mathfrak{N} \models \exists y\ \theta_\delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n}, y)$.

Definition 2.7. For every functional Δ_0-formula $\delta(x_1, \ldots, x_r, x, y)$ we define

$$\chi_\delta(x_1, \ldots, x_r, x) := \exists y \big(\theta_\delta(x_1, \ldots, x_r, x, y) \\ \wedge \forall y' \leqslant y \forall x' \leqslant y' (\theta_\delta(x_1, \ldots, x_r, x', y') \to x' \equiv x)\big). \qquad \dashv$$

The next lemma shows that χ_δ still defines the function f_δ introduced in Definition 2.6.

Lemma 2.8. *Let Δ_0-formula $\delta(x_1, \ldots, x_r, x_{r+1}, y)$ be functional and $n_1, \ldots, n_r, n \in \mathbb{N}$. Then*

$$f_\delta(n_1, \ldots, n_r) = n \quad \Longleftrightarrow \quad \mathfrak{N} \models \chi_\delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n})$$

Proof: Routine. □

Lemma 2.9. *Let Δ_0-formula $\delta(x_1, \ldots, x_r, x, y)$ be functional and $n_1, \ldots, n_r, n \in \mathbb{N}$.*

(C1) $\Phi_{\mathrm{PA}} \vdash \chi_\delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n})$ *if and only if* $\mathfrak{N} \models \chi_\delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n})$.

(C2) $\Phi_{\mathrm{PA}} \vdash \neg\chi_\delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n})$ *if and only if* $\mathfrak{N} \models \neg\chi_\delta(\bar{n}_1, \ldots, \bar{n}_r, n)$. $\dashv$

Proof: Observe that χ_δ in Definition 2.7 is a Σ_1-formula. Thus (C1) follows directly from Theorem 2.4. In (C2), the direction from left to right is trivial by $\mathfrak{N} \models \Phi_{\mathrm{PA}}$. For the converse, as δ is functional, there is a *unique* $n_{r+1} \in \mathbb{N}$ such that

$$\mathfrak{N} \models \exists y \big(\bar{n}_{r+1} \leqslant y \wedge \delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n}_{r+1}, y)\big).$$

Fix an arbitrary $m \in \mathbb{N}$ such that

$$\mathfrak{N} \models \bar{n}_{r+1} \leqslant \bar{m} \wedge \delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n}_{r+1}, \bar{m}), \tag{3}$$

i.e., $\mathfrak{N} \models \theta_\delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n}_{r+1}, \bar{m})$ by the definition (2) of θ_δ. As $\theta_\delta \in \Delta_0$, by Theorem 2.4,

$$\Phi_{\mathrm{PA}} \vdash \theta_\delta(\bar{n}_1, \ldots, \bar{n}_r, \bar{n}_{r+1}, \bar{m}).$$

Next, by the uniqueness of n_{r+1}

$$\mathfrak{N} \models \forall x \Big(\exists y \big(x \leqslant y \wedge \delta(\bar{n}_1, \ldots, \bar{n}_r, x, y)\big) \to x \equiv \bar{n}_{r+1}\Big),$$

which can be rewritten as

$$\mathfrak{N} \models \forall x' \forall y \Big(\big(x' \leqslant y \wedge \delta(\bar{n}_1, \ldots, \bar{n}_r, x', y)\big) \to x' \equiv \bar{n}_{r+1}\Big),$$

This implies that

$$\mathfrak{N} \models \forall y' \leqslant \bar{m} \forall x' \leqslant y' \Big(\big(x' \leqslant y' \wedge \delta(\bar{n}_1, \ldots, \bar{n}_r, x', y')\big) \to x' \equiv \bar{n}_{r+1}\Big), \tag{4}$$

i.e., $\mathfrak{N} \models \forall y' \leqslant \bar{m} \forall x' \leqslant y' \big(\theta_\delta(\bar{n}_1, \ldots, \bar{n}_r, x', y') \to x' \equiv \bar{n}_{r+1}\big)$. So again by Theorem 2.4,

$$\Phi_{\mathrm{PA}} \vdash \forall y' \leqslant \bar{m} \forall x' \leqslant y' \big(\theta_\delta(\bar{n}_1, \ldots, \bar{n}_r, x', y') \to x' \equiv \bar{n}_{r+1}\big).$$

By (3) and (4)

$$\mathfrak{N} \models \chi_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}_{r+1}).$$

As we have assumed $\mathfrak{N} \models \neg\chi_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n})$, it holds that

$$n \neq n_{r+1}.$$

Then Lemma 1.2 implies

$$\Phi_{PA} \vdash \neg\bar{n} \equiv \bar{n}_{r+1}.$$

Recall that our goal is to show

$$\Phi_{PA} \vdash \neg\chi_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}),$$

which is equivalent to show the inconsistency of

$$\Phi_{PA} \cup \big\{\chi_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n})\big\}.$$

This is further equivalent to the inconsistency of

$$\Phi_{PA} \cup \big\{\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, y), \forall y' \leqslant y \forall x' \leqslant y'(\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x', y') \to x' \equiv \bar{n})\big\}. \tag{5}$$

To ease presentation, let us list all the provable formulas obtained so far.

(P1) $\Phi_{PA} \vdash \theta_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}_{r+1}, \bar{m})$, in particular $\Phi_{PA} \vdash \bar{n}_{r+1} \leqslant \bar{m}$.

(P2) $\Phi_{PA} \vdash \forall y' \leqslant \bar{m} \forall x' \leqslant y'\big(\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x', y') \to x' \equiv \bar{n}_{r+1}\big)$.

(P3) $\Phi_{PA} \vdash \neg\bar{n} \equiv \bar{n}_{r+1}$.

Additionally we need

(P4) $\Phi_{PA} \vdash \bar{m} \leqslant y \vee y \leqslant \bar{m}$ by Lemma 1.2 (iii).

Then by instantiating y' by $\bar{m}$ and x' by $\bar{n}_{r+1}$, and by (P1) we obtain

$$\Phi_{PA} \cup \big\{\bar{m} \leqslant y, \forall y' \leqslant y \forall x' \leqslant y'(\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x', y') \to x' \equiv \bar{n})\big\} \vdash \bar{n}_{r+1} \equiv \bar{n}.$$

Together with (P3)

$$\Phi_{PA} \cup \big\{\bar{m} \leqslant y, \forall y' \leqslant y \forall x' \leqslant y'(\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x', y') \to x' \equiv \bar{n})\big\} \vdash \textsc{False}. \tag{6}$$

On the other hand, note

$$\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, y) \vdash \bar{n} \leqslant y.$$

Now by instantiating y' by y and x' by $\bar{n}$ in (P2) we get

$$\Phi_{PA} \cup \big\{y \leqslant \bar{m}, \theta_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, y)\big\} \vdash \bar{n} \equiv \bar{n}_{r+1}.$$

Therefore, (P3) implies

$$\Phi_{PA} \cup \big\{y \leqslant \bar{m}, \theta_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, y)\big\} \vdash \textsc{False}. \tag{7}$$

Combining (6), (7), and (P4), we conclude that (5) is inconsistent. □

Lemma 2.10. *Let Δ_0-formula $\delta(x_1, \dots, x_r, x_{r+1}, y)$ be functional and $n_1, \dots, n_r \in \mathbb{N}$. Then*

$$\Phi_{PA} \vdash \exists^{=1} x_{r+1} \chi_\delta(\bar{n}_1, \dots, \bar{n}_r, x_{r+1}).$$

Proof: Let $n \in \mathbb{N}$ be the unique natural number with

$$\mathfrak{N} \models \exists y\big(\bar{n} \leqslant y \land \delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, y)\big).$$

Choose an $m \in \mathbb{N}$ such that

$$\mathfrak{N} \models \bar{n} \leqslant \bar{m} \land \delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, \bar{m}).$$

As argued in the proof of Lemma 2.9 we can establish:

(U1) $\Phi_{\mathrm{PA}} \vdash \theta_\delta(\bar{n}_1, \dots, \bar{n}_r, \bar{n}, \bar{m})$. In particular, $\Phi_{\mathrm{PA}} \vdash \bar{n} \leqslant \bar{m}$.

(U2) $\Phi_{\mathrm{PA}} \vdash \forall y' \leqslant \bar{m} \forall x' \leqslant y'\big(\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x', y') \to x' \equiv \bar{n}\big)$.

(U3) $\Phi_{\mathrm{PA}} \vdash \bar{m} \leqslant y \lor y \leqslant \bar{m}$.

We show that

$$\Phi_{\mathrm{PA}} \vdash \forall x\big(\chi_\delta(\bar{n}_1, \dots, \bar{n}_r, x) \to x \equiv \bar{n}\big). \tag{8}$$

Then the result follows immediately. Observe that (8) is equivalent to

$$\Phi_{\mathrm{PA}} \cup \big\{\chi_\delta(\bar{n}_1, \dots, \bar{n}_r, x)\big\} \vdash x \equiv \bar{n},$$

i.e.,

$$\Phi_{\mathrm{PA}} \cup \big\{\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x, y), \forall y' \leqslant y \forall x' \leqslant y'(\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x', y') \to x' \equiv x)\big\} \vdash x \equiv \bar{n}. \tag{9}$$

By instantiating y' by $\bar{m}$ and x' by $\bar{n}$, and by (U1) we obtain

$$\Phi_{\mathrm{PA}} \cup \big\{\bar{m} \leqslant y, \forall y' \leqslant y \forall x' \leqslant y'(\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x', y') \to x' \equiv x)\big\} \vdash x \equiv \bar{n}. \tag{10}$$

Next, observe

$$\theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x, y) \vdash x \leqslant y.$$

Now by instantiating y' by y and x' by x in (U2) we get

$$\Phi_{\mathrm{PA}} \cup \big\{y \leqslant \bar{m}, \theta_\delta(\bar{n}_1, \dots, \bar{n}_r, x, y)\big\} \vdash x \equiv \bar{n}. \tag{11}$$

With (10), (11), and (U3) we have shown (9). □

Combing Lemma 2.9, Lemma 2.10, and Lemma 2.8:

Theorem 2.11. *Let Δ_0-formula $\delta(x_1, \dots, x_r, x_{r+1}, y)$ be functional. Then f_δ is representable in Φ_{PA} by the formula χ_δ.*

3. Representation in Φ_{PA}

We have shown:

Theorem 3.1. Th($\mathfrak{N}$) *allows representations. More precisely:*

(i) For any R-decidable relation $\mathscr{R} \subseteq \mathbb{N}^r$ there is an $L^{S_{\mathrm{ar}}}$-formula $\varphi(v_0, \dots, v_{r-1})$ such that for all $n_0, \dots, n_{r-1} \in \mathbb{N}$

$$\begin{aligned}(n_0, \dots, n_{r-1}) \in \mathscr{R} &\implies \mathfrak{N} \models \varphi(\bar{n}_0, \dots, \bar{n}_{r-1}),\\ (n_0, \dots, n_{r-1}) \notin \mathscr{R} &\implies \mathfrak{N} \models \neg\varphi(\bar{n}_0, \dots, \bar{n}_{r-1}).\end{aligned}$$

(ii) For any R-computable function $f : \mathbb{N}^r \to \mathbb{N}$ there is an $L^{S_{ar}}$-formula $\varphi(v_0, \dots, v_{r-1}, v_r)$ such that for all $n_0, \dots, n_{r-1}, n_r \in \mathbb{N}$

$$f(n_0, \dots, n_{r-1}) = n_r \implies \mathfrak{N} \models \varphi(\bar{n}_0, \dots, \bar{n}_{r-1}, \bar{n}_r),$$
$$f(n_0, \dots, n_{r-1}) \neq n_r \implies \mathfrak{N} \models \neg\varphi(\bar{n}_0, \dots, \bar{n}_{r-1}, \bar{n}_r).$$

Moreover,

$$\mathfrak{N} \models \exists^{=1} v_r \ \varphi(\bar{n}_0, \dots, \bar{n}_{r-1}, v_r).$$

Furthermore, a careful inspection shows that in both cases the formula φ can be chosen in Σ_1. ⊣

Lemma 3.2. *Let $f : \mathbb{N}^r \to \mathbb{N}$ be R-computable. Then f is representable in Φ_{PA}.*

Proof: By Theorem 3.1 there is a Σ_1-formula φ representing f in $\mathfrak{N}$. We assume

$$\varphi(v_0, \dots, v_{r-1}, v_r) = \exists y_1 \dots \exists y_k \ \psi(v_0, \dots, v_{r-1}, v_r, y_1, \dots, y_k)$$

with $\psi \in \Delta_0$. We define

$$\delta(v_0, \dots, v_{r-1}, v_r, y) := \exists y_1 \leqslant y \dots \exists y_k \leqslant y \ \psi(v_0, \dots, v_{r-1}, v_r, y_1, \dots, y_k)$$

which is a Δ_0-formula.

Claim 1. δ is functional.

Proof of the claim. Let $n_0, \dots, n_{r-1} \in \mathbb{N}$. Define

$$n := f(n_0, \dots, n_{r-1}).$$

Since φ represents f in $\mathfrak{N}$, there are $m_1, \dots, m_k \in \mathbb{N}$ such that

$$\mathfrak{N} \models \psi(\bar{n}_0, \dots, \bar{n}_r, \bar{n}, \bar{m}_1, \dots, \bar{m}_k).$$

Choose

$$p := \max\{n, m_1, \dots, m_k\}.$$

Then

$$\mathfrak{N} \models \bar{n} \leqslant \bar{p} \wedge \big(\bar{m}_1 \leqslant \bar{p} \wedge \dots \wedge \bar{m}_k \leqslant \bar{p} \wedge \psi(\bar{n}_0, \dots, \bar{n}_r, \bar{n}, \bar{m}_1, \dots, \bar{m}_k)\big).$$

Hence,

$$\mathfrak{N} \models \bar{n} \leqslant \bar{p} \wedge \big(\exists y_1 \leqslant \bar{p} \dots \exists y_k \leqslant \bar{p} \ \psi(\bar{n}_0, \dots, \bar{n}_r, \bar{n}, y_1, \dots, y_k)\big).$$

That is

$$\mathfrak{N} \models \bar{n} \leqslant \bar{p} \wedge \delta(\bar{n}_0, \dots, \bar{n}_r, \bar{n}, \bar{p}).$$

Hence,

$$\mathfrak{N} \models \exists y \big(\bar{n} \leqslant y \wedge \delta(\bar{n}_0, \dots, \bar{n}_r, \bar{n}, y)\big).$$

To see the uniqueness, let $n' \in \mathbb{N}$ satisfy

$$\mathfrak{N} \models \exists y \big(\bar{n}' \leqslant y \wedge \delta(\bar{n}_0, \dots, \bar{n}_r, \bar{n}', y)\big).$$

By the definition of δ

$$\mathfrak{N} \models \exists y_1 \dots \exists y_k \ \psi(\bar{n}_0, \dots, \bar{n}_{r-1}, \bar{n}', y_1, \dots, y_k)$$

Equivalently,

$$\mathfrak{N} \models \varphi(\bar{n}_0, \dots, \bar{n}_{r-1}, \bar{n}').$$

Since φ represents the function f, we conclude that

$$n' = n.$$

This proves the claim. $\dashv$

Claim 2. Let $n_0, \ldots, n_{r-1}, n \in \mathbb{R}$. Then

$$f(n_0, \ldots, n_{r-1}) = n \quad \Longleftrightarrow \quad \mathfrak{N} \models \exists y \big(\bar{n} \leqslant y \wedge \delta(\bar{n}_0, \ldots, \bar{n}_{r-1}, \bar{n})\big).$$

Equivalently, since δ is functional by Claim 1,

$$f(n_0, \ldots, n_{r-1}) = n \quad \Longleftrightarrow \quad f_\delta(n_0, \ldots, n_{r-1}) = n.$$

Proof of the claim. we deduce

$$\begin{aligned} & f(n_0, \ldots, n_{r-1}) = n \\ & \qquad \Longleftrightarrow \mathfrak{N} \models \varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}, \bar{n}) \\ & \qquad \Longleftrightarrow \mathfrak{N} \models \exists y_1 \ldots \exists y_k \, \psi(\bar{n}_0, \ldots, \bar{n}_{r-1}, \bar{n}, y_1, \ldots, y_k) \\ & \qquad \Longleftrightarrow \mathfrak{N} \models \exists y \big(\bar{n} \leqslant y \wedge \exists y_1 \leqslant y \ldots \exists y_k \leqslant y \; \psi(\bar{n}_0, \ldots, \bar{n}_{r-1}, \bar{n}, y_1, \ldots, y_k)\big) \\ & \qquad \Longleftrightarrow \mathfrak{N} \models \exists y \big(\bar{n} \leqslant y \wedge \delta(\bar{n}_0, \ldots, \bar{n}_{r-1}, \bar{n}, y)\big) \end{aligned}$$

$\dashv$

By Theorem 2.11, Claim 2 implies that χ_δ represents the function f in Φ_{PA}. □

Lemma 3.3. *Let $\mathscr{R} \subseteq \mathbb{N}^r$ be R-decidable. Then $\mathscr{R}$ is representable in Φ_{PA}.*

Proof: We define the characteristic function $f_\mathscr{R}$ for $\mathscr{R}$ as follows. For every $n_0, \ldots, n_{r-1} \in \mathbb{N}$ let

$$f_\mathscr{R}(n_0, \ldots, n_{r-1}) := \begin{cases} 1 & \text{if } (n_0, \ldots, n_{r-1}) \in \mathscr{R} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f_\mathscr{R}$ is R-computable. So by Lemma 3.2 there is a formula $\psi(\nu_0, \ldots, \nu_{r-1}, \nu)$ representing $f_\mathscr{R}$ in Φ_{PA}. Define

$$\varphi(\nu_0, \ldots, \nu_{r-1}) := \psi(\nu_0, \ldots, \nu_{r-1}, 1).$$

Now assume $(n_0, \ldots, n_{r-1}) \in \mathscr{R}$, then $f_\mathscr{R}(n_0, \ldots, n_{r-1}) = 1$. It follows that

$$\Phi_{\text{PA}} \vdash \psi(\bar{n}_0, \ldots, \bar{n}_{r-1}, 1).$$

Thereby

$$\Phi_{\text{PA}} \vdash \varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}).$$

Otherwise, $(n_0, \ldots, n_{r-1}) \notin \mathscr{R}$, then $f_\mathscr{R}(n_0, \ldots, n_{r-1}) \neq 1$. Hence

$$\Phi_{\text{PA}} \vdash \neg\psi(\bar{n}_0, \ldots, \bar{n}_{r-1}, 1),$$

i.e.,

$$\Phi_{\text{PA}} \vdash \neg\varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}).$$

This shows that $\mathscr{R}$ is representable in Φ_{PA} by the formula φ. □

By Lemma 3.2 and Lemma 3.3:

Theorem 3.4. Φ_{PA} *allows representations.*