A Faster Deterministic Algorithm For 3-SAT

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Overview

1. The SAT problem and some algorithms
2. The `searchball` algorithm
3. Improving `searchball`

SAT – What Is It About?

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

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Assignment α: $x = 0, y = 1, z = 1$

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Assignment α: $x = 0, y = 1, z = 1$

Assignment β: $x = 1, y = 1, z = 0$

SAT – What Is It About?

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

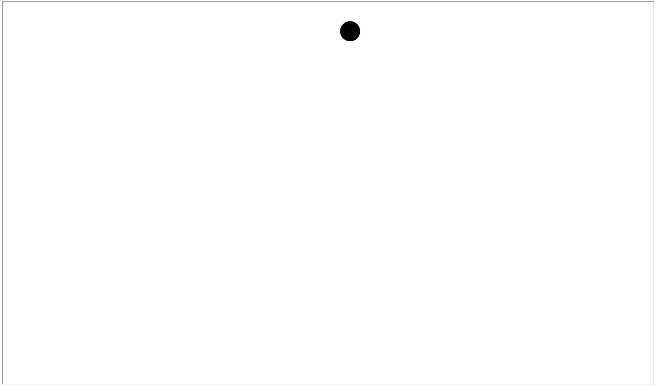
Assignment α: $x = 0, y = 1, z = 1$

Assignment β: $x = 1, y = 1, z = 0$

β satisfies the formula, α does not.

A Recursive Algorithm

$$\{\{x, y, z\}, \{x, \bar{y}, z\}, \{x, y, \bar{z}\}, \{\bar{x}, y, z\}\}$$



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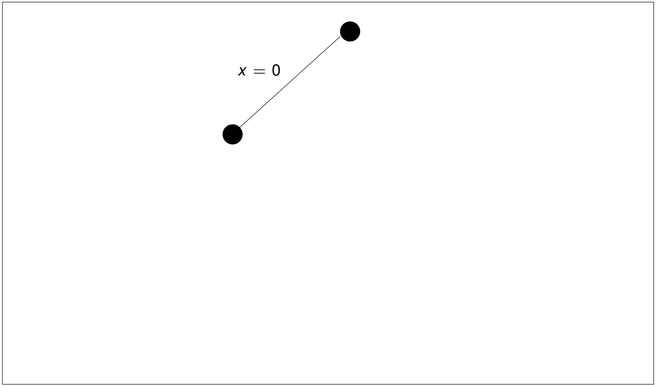

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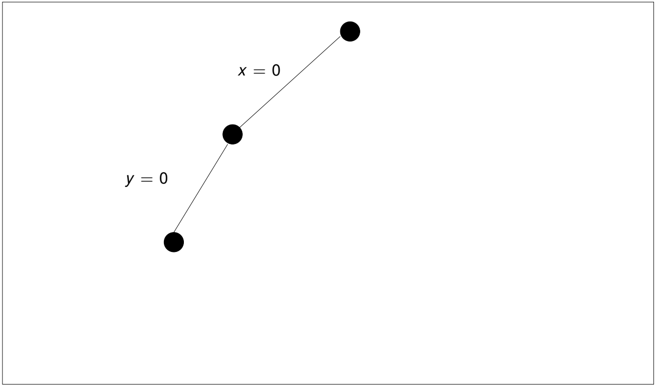

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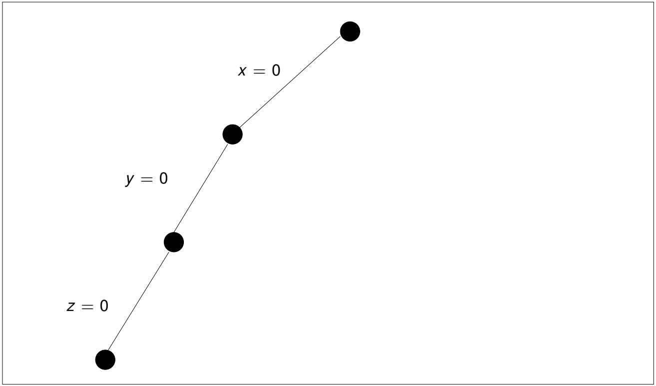

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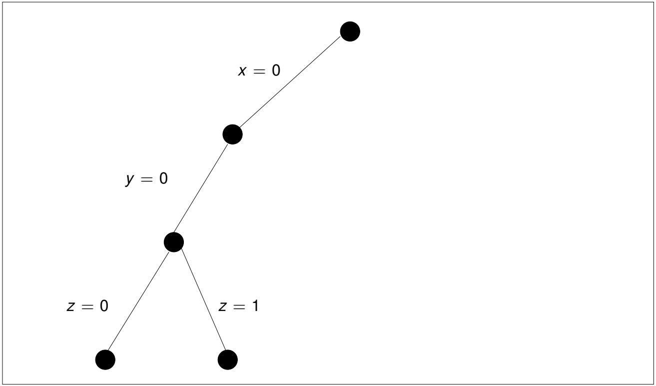

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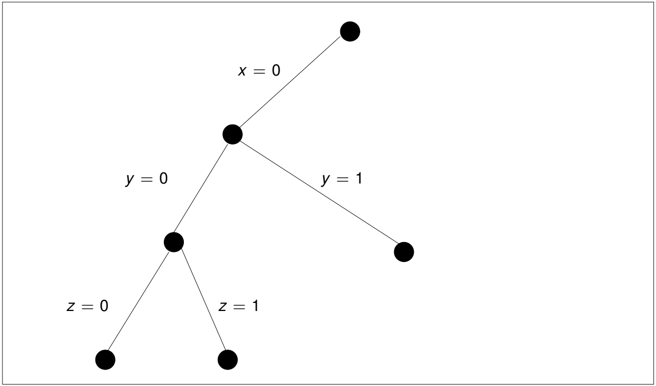

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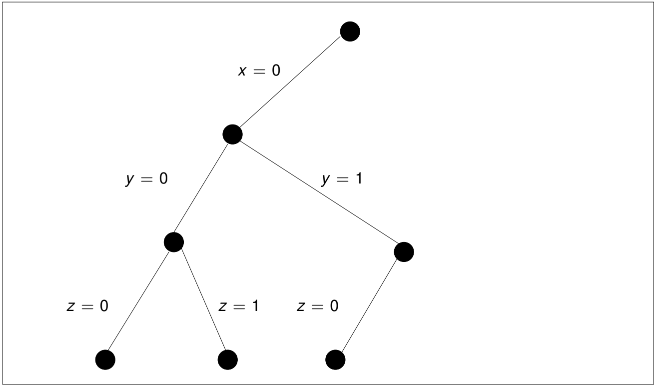

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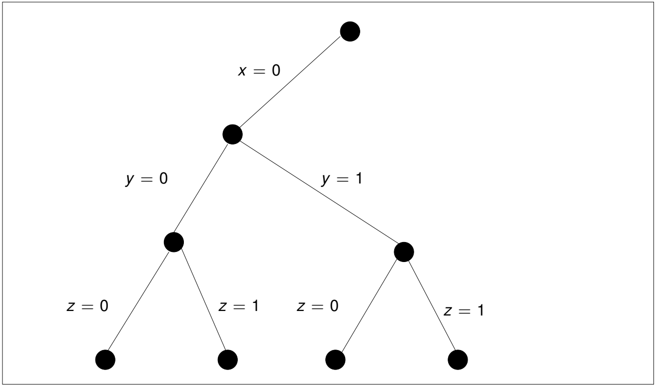

Satisfying Assignment: $x = 0, y = 1, z = 1$

Algorithms for 3-SAT — A Very Incomplete Overview

2^n	What I just showed you		deterministic
1.618^n	Monien and Speckenmeyer	1985	deterministic
1.588^n	Paturi, Pudlák and Zane	1997	randomized
1.5^n	Dantsin et. al.	2002	deterministic
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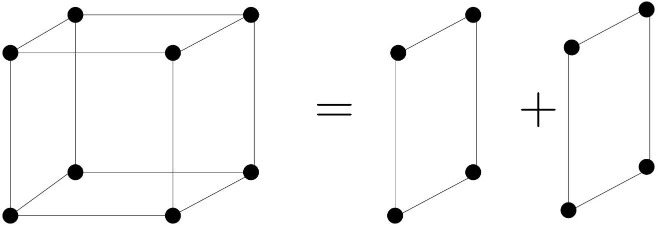
Visualization of Trivial Branching Algorithm

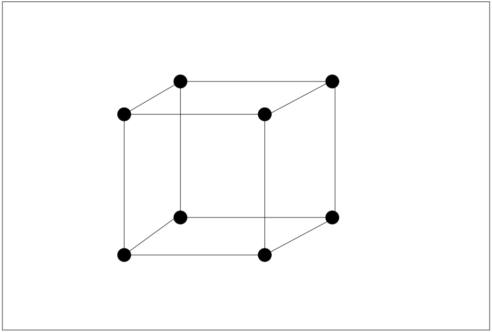
Set of truth assignments $= \{0, 1\}^n = n$-dimensional cube



Decompose into two $(n-1)$-dimensional cubes

Idea of Local Search

Cover $\{0,1\}^n$ by several Hamming balls



$$B_r(x) = \left\{ y \in \{0,1\}^n \middle| d_H(x,y) \le r \right\}$$

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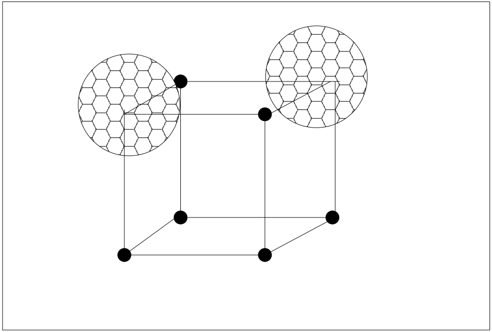

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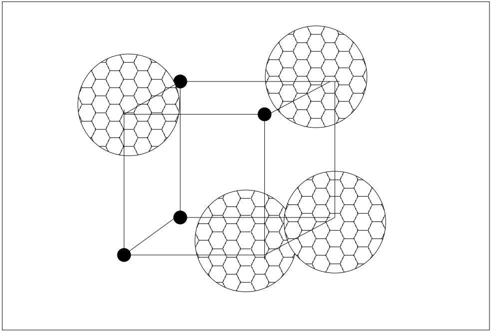

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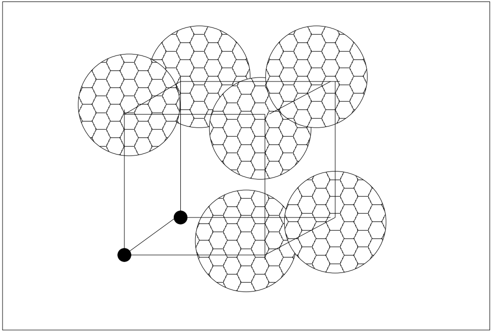

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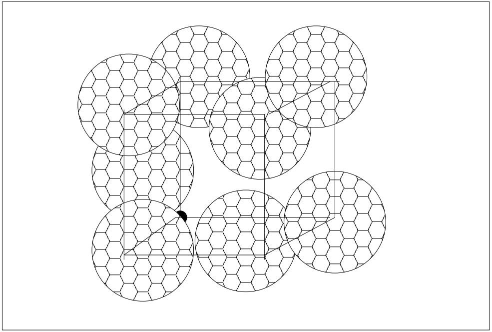

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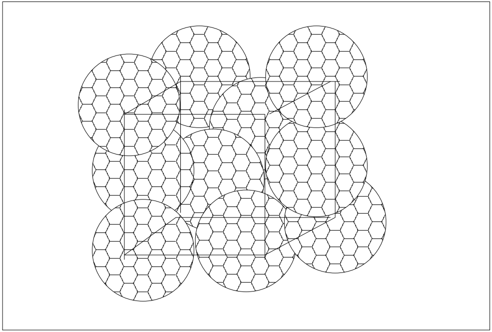
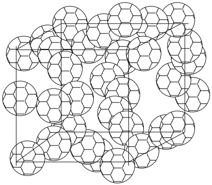
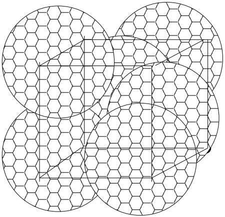


$$B_r(x) = \left\{ y \in \{0,1\}^n \middle| d_H(x,y) \le r \right\}$$

Choosing the Optimal Radius

Should we take many small balls or a few large ones?





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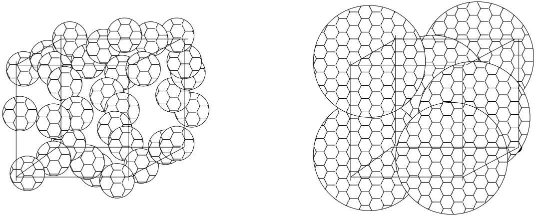

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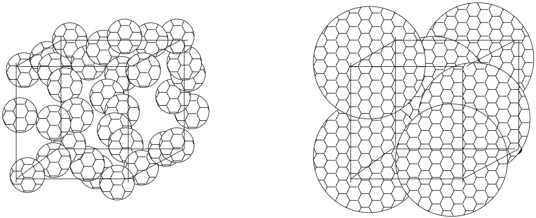

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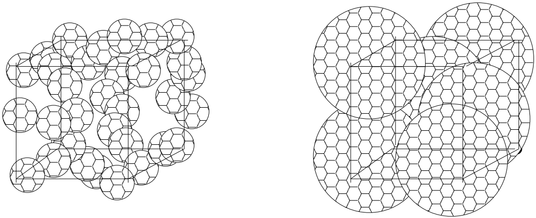

- Suppose we need $\mathcal{O}(a^r)$ steps per ball
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- $\mathcal{O}\left(\left(2 - \frac{2}{a+1}\right)^n\right)$ steps for all balls

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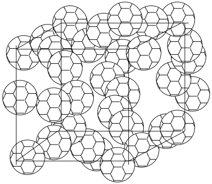

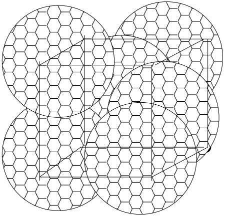

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a^r	$\left(2 - \frac{2}{a+1}\right)^n$
3^r	1.5^n
2.848^r	1.481^n
2.792^r	1.473^n
2.74^r	1.465^n

2. The `searchball` Algorithm

Defintion of “Setting a variable”

$$F = \{\{x, y, z\}, \{x, \bar{y}, z\}, \{x, y, \bar{z}\}, \{\bar{x}, y, z\}\}$$

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$$\{\{\color{red}{x}, y, z\}, \{\color{red}{x}, \bar{y}, z\}, \{\color{red}{x}, y, \bar{z}\}, \{\color{green}{\bar{x}}, y, z\}\}$$

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Set variable x to 0

$$\{\{x, y, z\}, \{x, \bar{y}, z\}, \{x, y, \bar{z}\}, \{\bar{x}, y, z\}\}$$

$$F^{[x=0]} = \{\{ \quad y, z\}, \{ \quad \bar{y}, z\}, \{ \quad y, \bar{z}\} \qquad \qquad \}$$

The Searchball Algorithm

`searchball`(Formula F, depth r, assignment α)

if α satisfies F **then**
    return `true`
else if $\square \in F$ or $r \leq 0$ **then**
    return `false`
else
    pick some unsatisfied clause $\{u_1, \ldots, u_\ell\} \in F$
    return $\bigvee_{i=1}^{\ell}$ `searchball`$(F^{[u_i \mapsto 1]}, r-1)$
end if

The Searchball Algorithm

searchball(Formula F, depth r, assignment α)

```
1: if α satisfies F then
2:     return true
3: else if □ ∈ F or r ≤ 0 then
4:     return false
5: else
6:     pick some unsatisfied clause {u_1, ..., u_ℓ} ∈ F
7:     return ⋁_{i=1}^{ℓ} searchball(F^[u_i ↦ 1], r − 1)
8: end if
```

From now on, $\alpha = (1, \ldots, 1)$

Visualization of Searchball

Radius: $r = 2$ Center of ball: $\alpha = (1, 1, 1, 1)$

$$\{\{\bar{x}, \bar{y}, \bar{z}\}, \{\bar{y}, \bar{z}, \bar{w}\}, \{x, y, \bar{z}\}, \{x, \bar{y}, z\}, \{x, \bar{y}, w\}, \{y, \bar{z}, \bar{w}\}, \{\bar{x}, y, z\}\}$$

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$\{\bar{x}, \bar{y}, \bar{z}\}$

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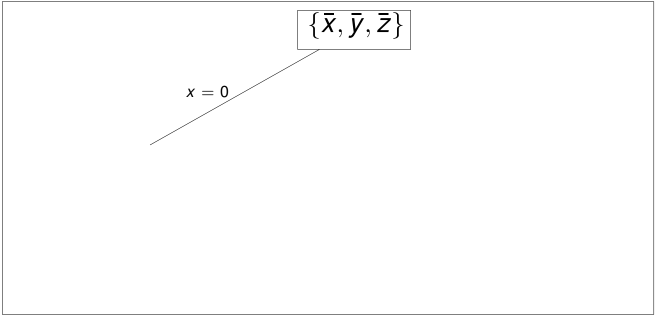

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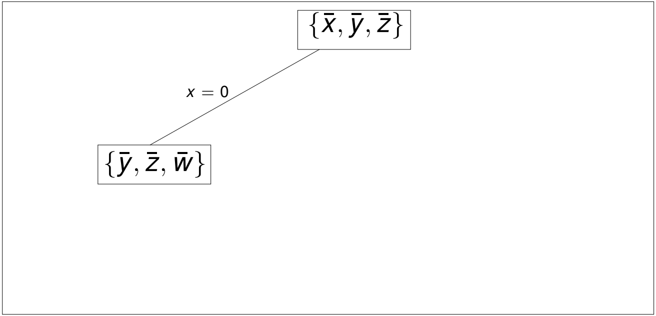

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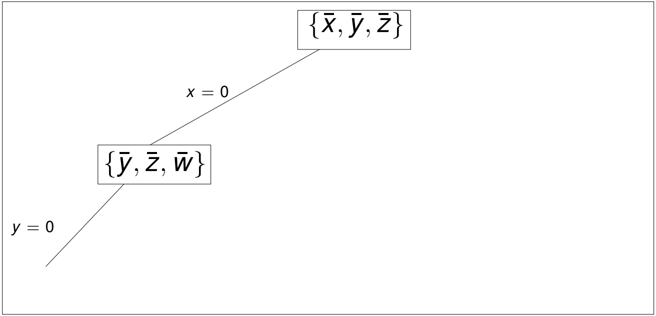

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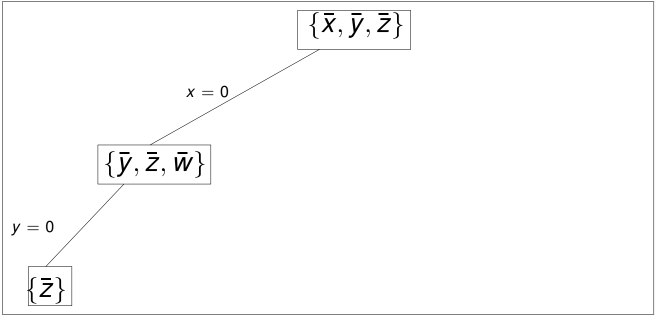

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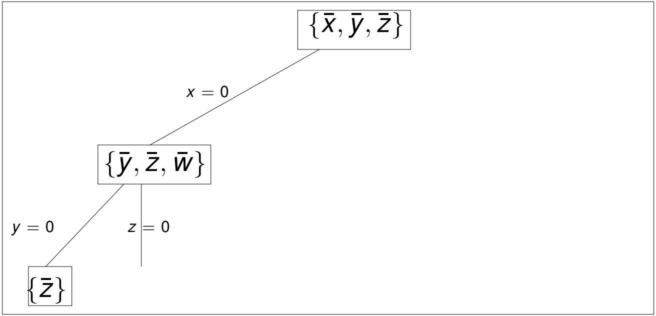

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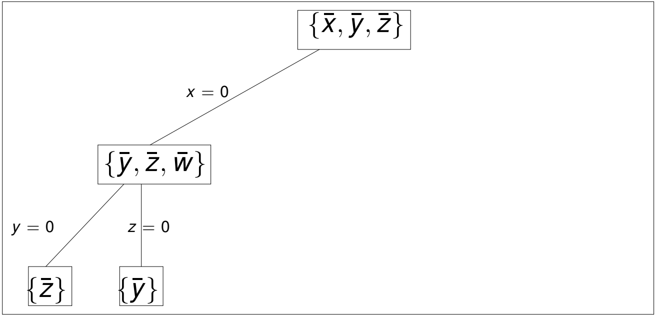

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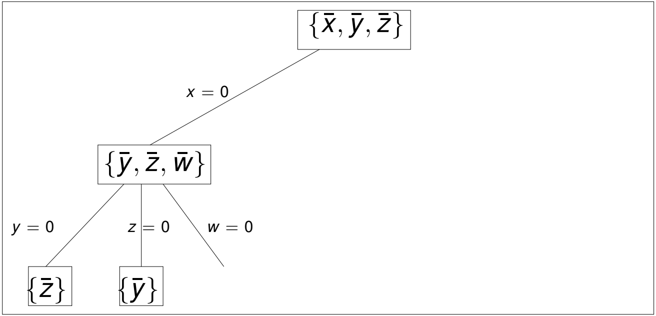

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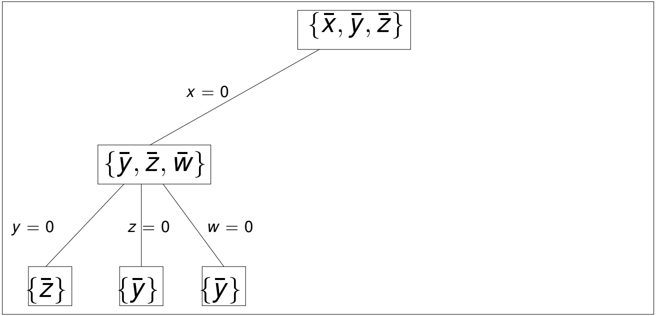

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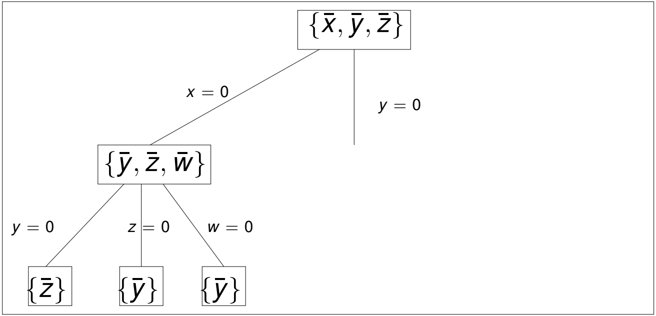

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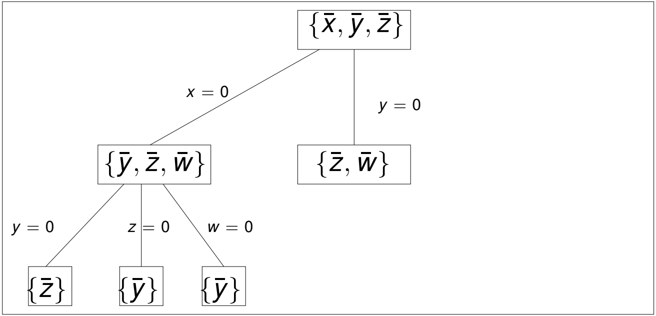

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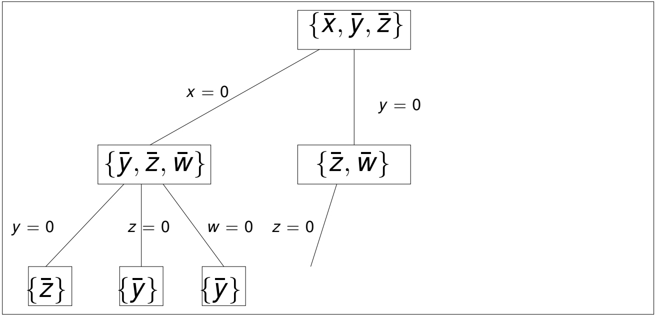

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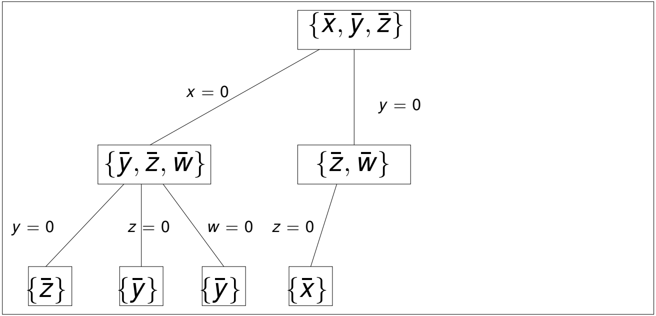

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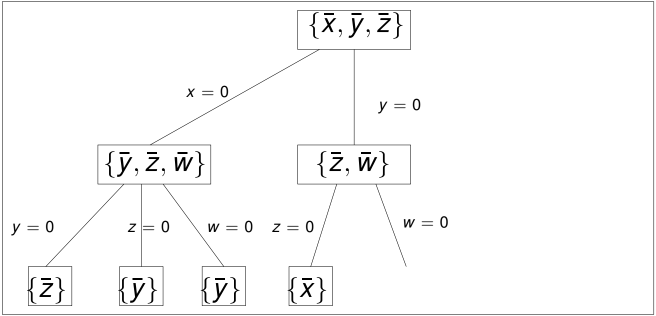
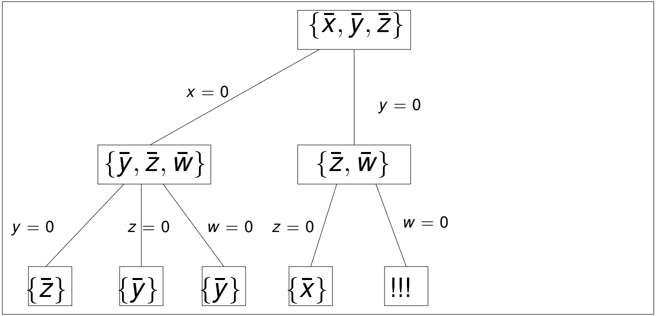


Visualization of Searchball

Radius: $r = 2$ Center of ball: $\alpha = (1, 1, 1, 1)$

$\{\{\bar{x}, \bar{y}, \bar{z}\}, \{\bar{y}, \bar{z}, \bar{w}\}, \{x, y, \bar{z}\}, \{x, \bar{y}, z\}, \{x, \bar{y}, w\}, \{y, \bar{z}, \bar{w}\}, \{\bar{x}, y, z\}\}$



Satisfying Assignment: $x = 1, y = 0, z = 1, w = 0$

Running Time of Searchball

How many leaves does this search tree have?

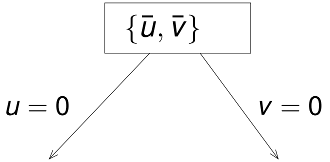
- For a 3-CNF formula, at most 3^r leaves
- But: Not every node has 3 children!
- Try to generate small unsatisfied clauses
- Four quite simple rules

3. Improving `searchball`

Four Rules

Rule 1

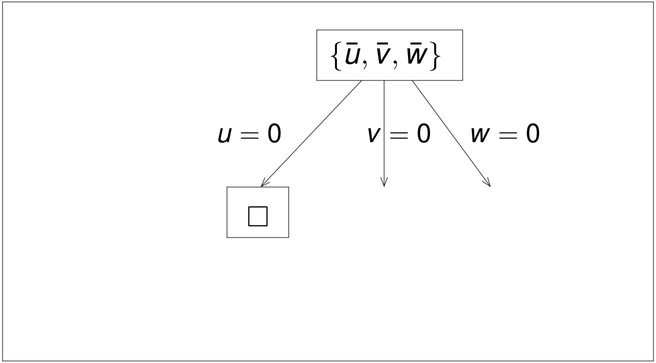
If $\{\bar{u}, \bar{v}\} \in F$ (or $\{\bar{u}\} \in F$, or $\square \in F$:



$$L(r) \leq 2L(r-1) \quad \Longrightarrow \quad L(r) \leq 2^r$$

Rule 2

If $\{u\}, \{\bar{u}, \bar{v}, \bar{w}\} \in F$:

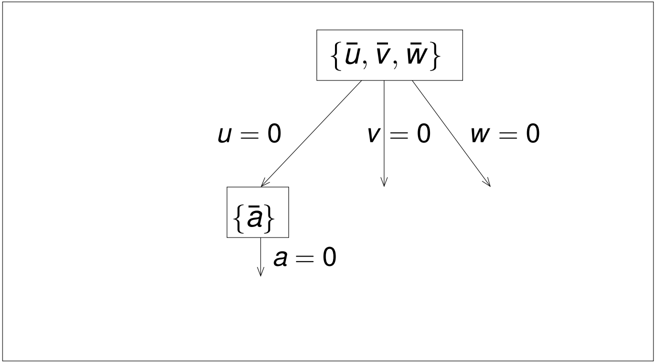


$$L(r) \leq 2L(r-1) \quad \Longrightarrow \quad L(r) \leq 2^r$$

Four Rules

Rule 3

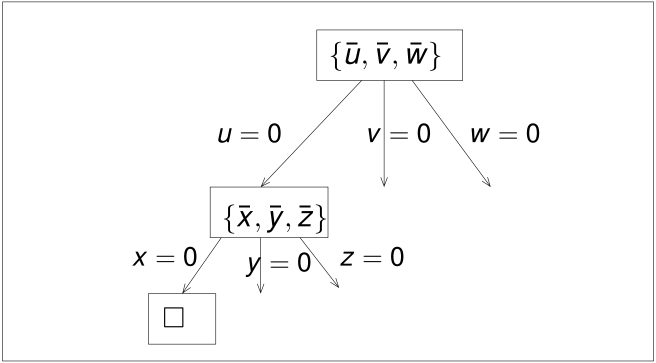
If $\{u, \bar{a}\}, \{\bar{u}, \bar{v}, \bar{w}\} \in F$:



$$L(r) \leq 2L(r-1) + L(r-2) \quad \Longrightarrow \quad L(r) \leq 2.414^r$$

Rule 4

If $\{u, x\}, \{\bar{u}, \bar{v}, \bar{w}\}, \{\bar{x}, \bar{y}, \bar{z}\} \in F$:

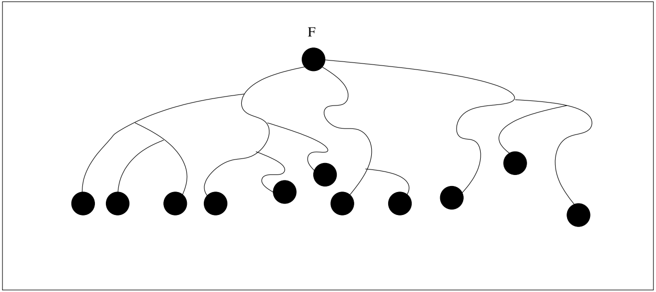


$$L(r) \leq 2L(r-1) + 2L(r-2) \quad \Longrightarrow \quad L(r) \leq 2.73^r$$

Reduced Formulas

Otherwise: *F* is a *reduced* formula

Run a “preliminary search”:



Reduced Formulas

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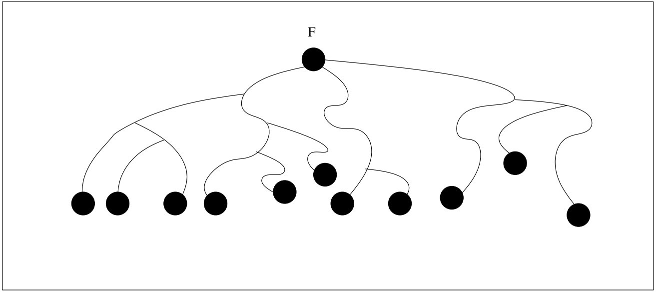

1. Path $\equiv$ set of variables.

Reduced Formulas

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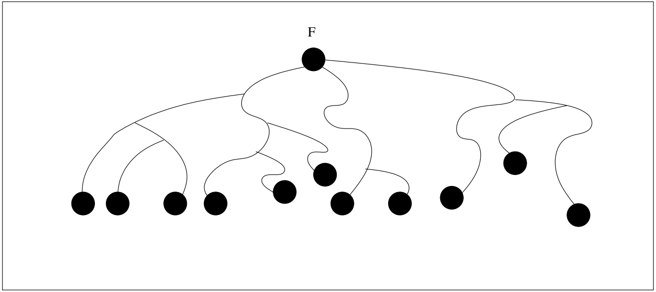
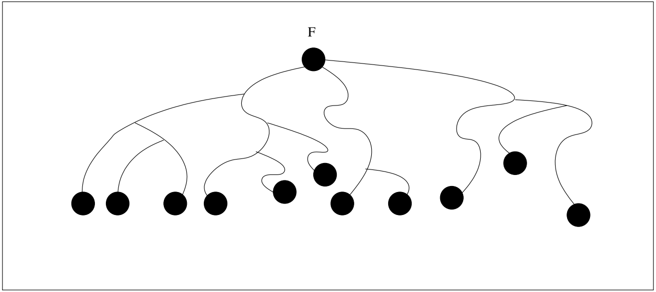


1. Path $\equiv$ set of variables.
2. Only allow certain “nice” sets.

Otherwise: *F* is a *reduced* formula

Run a “preliminary search”:

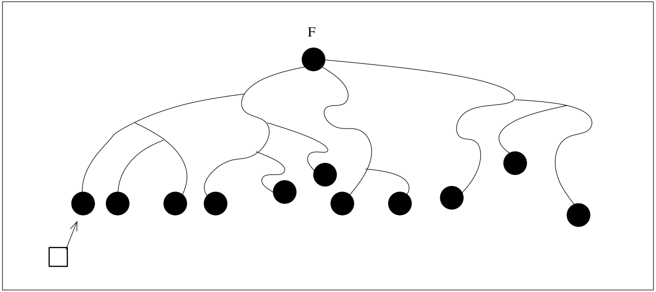


1. Path $\equiv$ set of variables.
2. Only allow certain “nice” sets.
3. Abort search at reduced formulas.

Reduced Formulas

Otherwise: F is a *reduced* formula

Run a "preliminary search":



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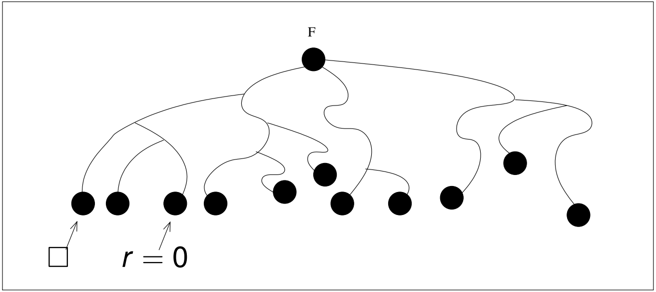

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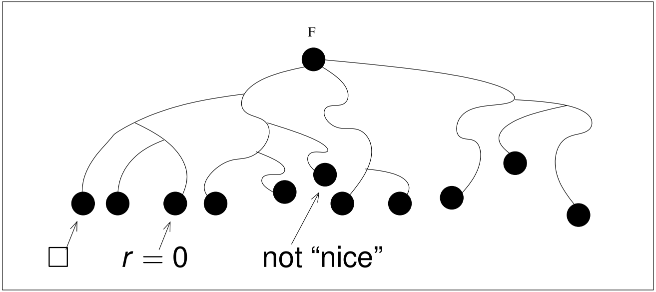

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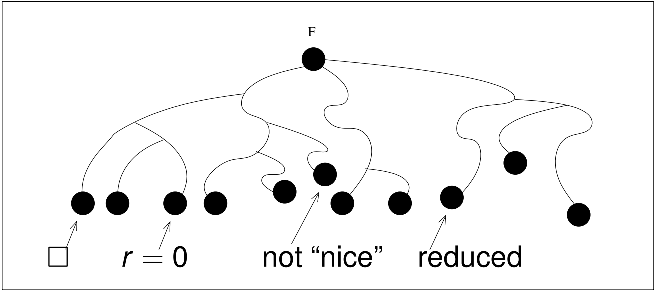

1. Path $\equiv$ set of variables.
2. Only allow certain "nice" sets.
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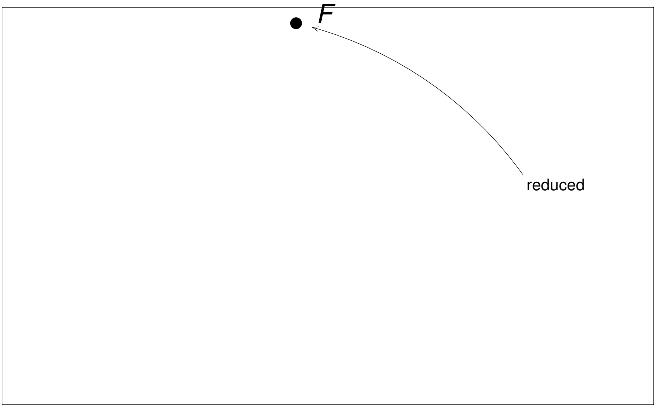
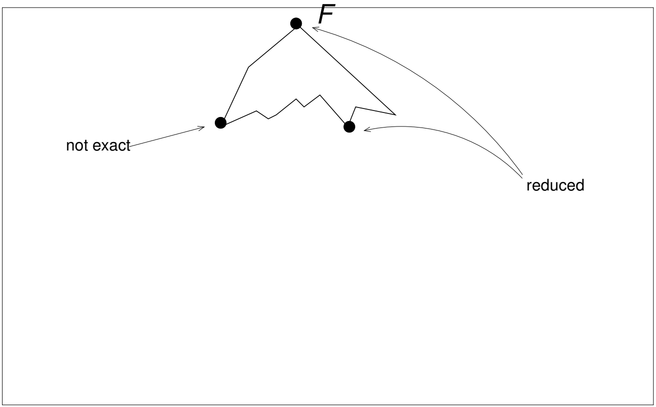
Repeat Preliminary Search:

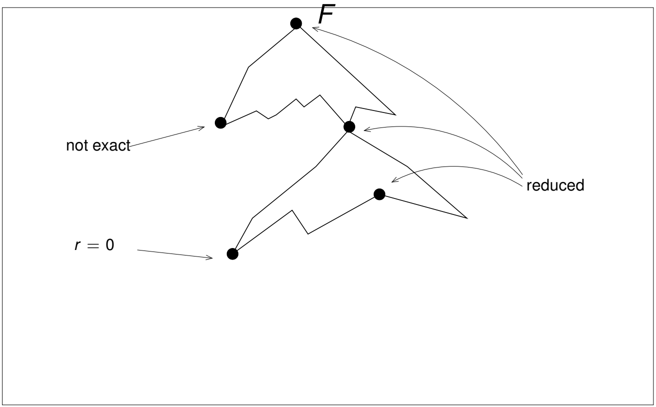


Dealing With Reduced Formulas

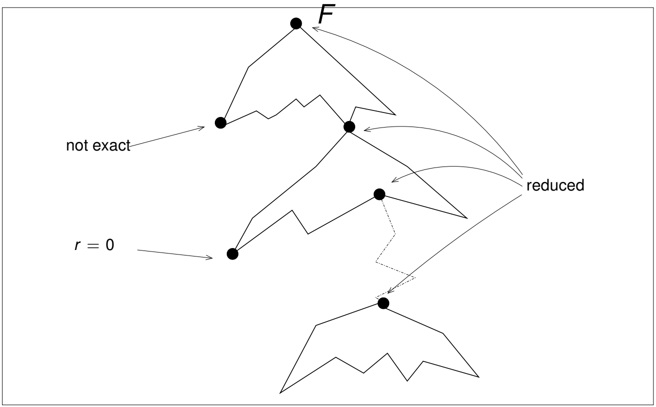
Repeat Preliminary Search:



Repeat Preliminary Search:



Repeat Preliminary Search:



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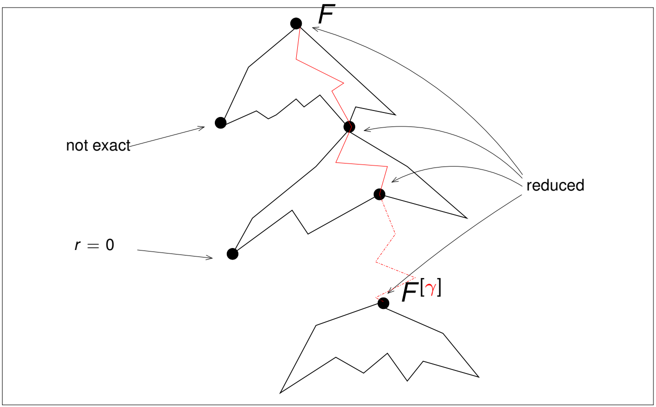

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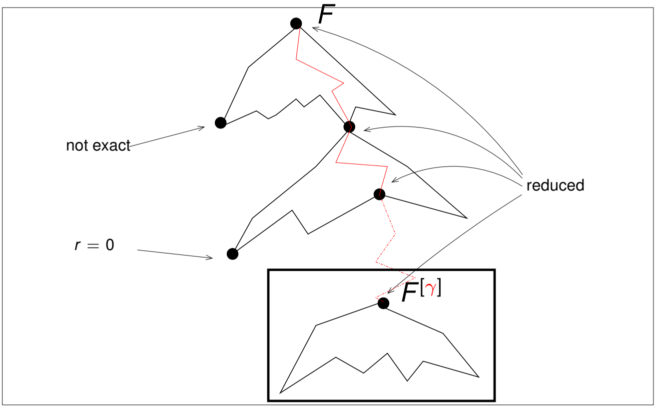
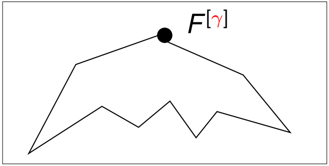


The Guide Formula

Focus on $F^{[\gamma]}$



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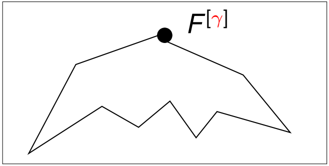
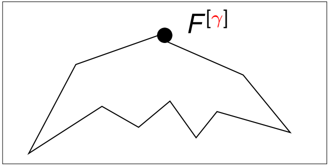


- No reduced formulas in this tree.

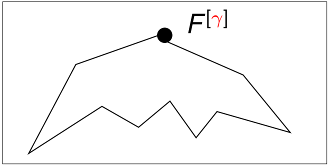
Focus on $F^{[\gamma]}$



- No reduced formulas in this tree.
- Use the same tree for F!

The Guide Formula

Focus on $F^{[\gamma]}$



- No reduced formulas in this tree.
- Use the same tree for F!
- Use $F^{[\gamma]}$ as a “guide” for F.

Summary of the Algorithm

- Rule 1–4 have good running time.
- We use a poor branching on reduced formulas.
- If we branch as the “guide” tells us, we will not encounter too many reduced formulas.
- Rule 4 dominates the running time:
  $L(r) \leq 2L(r-1) + 2L(r-2)$.
  $L(r) \in \mathcal{O}(2.74^r)$

Conclusions

What did we achieve?

- Search $B_r(1, \ldots, 1)$ in $\mathcal{O}(2.74^r)$ steps.
- Solve 3-SAT in $\mathcal{O}(1.465^n)$ steps.

SO WHAT???

- Find a simpler way to describe the algorithm ("What is really going on?")
- Apply the idea to other problems?

Muito obrigado pela sua atenção!!!