Mathematical Foundations of Computer Science

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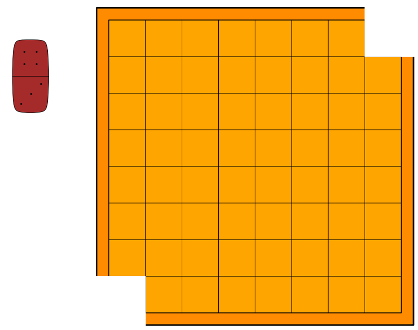
- Homework assignment published on Monday, 2017-02-20
- Work on it and submit questions by Wednesday, 2017-02-22, 18:00 by email to dominik.scheder@gmail.com
- We will discuss some problems on Thursday, 2017-02-23.
- Submit your solution by Sunday, 2017-02-26 to me and TA.
- We will give you feedback, you will have an opportunity to revise your solution before we will grade it.

1 Broken Chessboard and Jumping With Coins

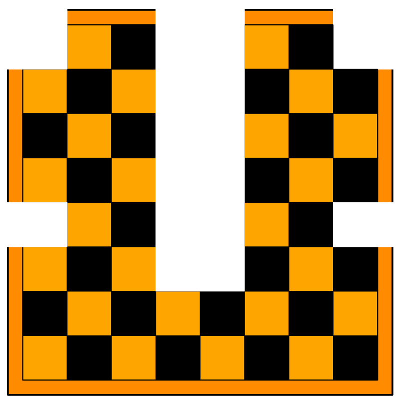
1.1 Tiling a Damaged Checkerboard

In the video lecture you have seen a proof that one cannot tile the "damaged" 8×8 checkerboard with domino stones:



Exercise 1.1. Re-write the proof in your own way, using simple English sentences.

Exercise 1.2. Look at the seriously damaged 8×8 checkerboard. For convenience I already colored it black and white (or rather black and beige):



Try to tile it with domino stones and you will fail. However, since there are 24 black and 24 beige squares, the simple argument from the lecture will fail.

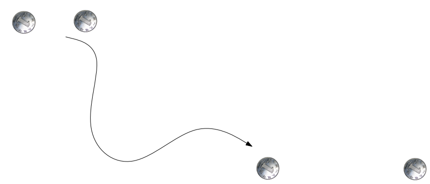
Prove that the above board cannot be tiled. Try to find a short and simple argument!

1.2 Jumping with Coins

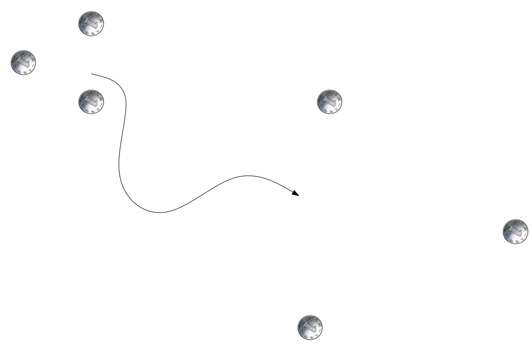
This is simply a reminder of the exercises I posed in the video lecture. For details, refer to the video.

Remark. The following exercises are of different levels of difficulty, but they all have a very simple proof (altough the proof might not be easy to find).

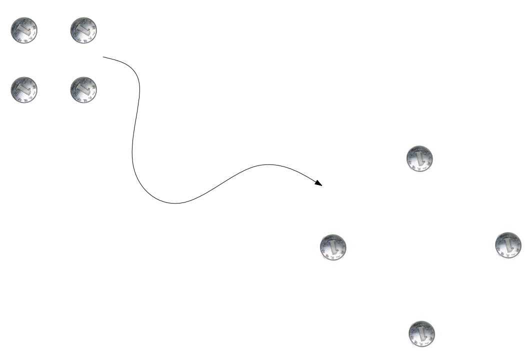
Exercise 1.3. You jump around with two coins. Show that you cannot increase the distance between the two coins.



Exercise 1.4. You jump around with three coins. Show that you cannot start with an equilateral triangle and end up with a bigger equilateral triangle. Give a simple proof!



You jump around with four coins which in the beginning form a square of side length 1.



Exercise 1.5. Show that you cannot form a square of side length 2.

Exercise 1.6. Show that you cannot achieve a position in which two coins are at the same position.

Exercise 1.7. Show that you cannot form a larger square.

***Exercise 1.8.** We can denote a "coin configuration" as a tuple (a, b, c, d) where $a, b, c, d \in \mathbb{R}^2$. Let $U = (a, b, c, d)$ and $V = (a', b', c', d')$ be two coin configurations. We write $U \to V$ if we can go from U to V by jumping and $U \not\to V$ if not. For example,

$$((0,0),(1,0),(0,1),(1,1)) \to ((0,1),(0,2),(1,0),(1,1)) \text{ but}$$
$$((0,0),(1,0),(0,1),(1,1)) \not\to ((0,0),(2,0),(0,2),(2,2)) \text{ and certainly}$$
$$((0,0),(1,0),(0,1),(1,1)) \not\to ((0,0.5),(\pi,0),(0,\sqrt{2}),(\sqrt{3},\sqrt{5}))\ .$$

Can you find a general principle by which we can determine whether $U \to V$? Formally, can you find a *simple and efficient* algorithm which, on input U, V, determines whether $U \to V$ holds?

2 Exclusion-Inclusion

2.1 Sets

Exercise 2.1. Let A, B, C be finite sets.

1. Prove that $|A \cup B| = |A| + |B| - |A \cap B|$.
2. What about $|A \cup B \cup C|$? Find a formula in terms of pairwise and three-wise intersections.
3. What about $|A \cup B \cup C \cup D|$? Find a formula in terms of pairwise, three-wise, and four-wise intersections.

Exercise 2.2. [The Exclusion-Inclusion Formula] Maybe you have noticed a pattern. Find a general formula, i.e., for $|A_1 \cup \cdots \cup A_n|$ in terms of the size of intersections $A_I := \bigcap_{i \in I} A_i$.

Exercise 2.3. Justify the formula you found in the previous exercise. **Hint.** There is a proof using induction on n. **Hint.** There is a proof that does not need induction on n.

3 Feasible Intersection Patterns

Exercise 3.1. Find sets A_1, A_2, A_3, A_4 such that all pairwise intersections have size 3 and all three-wise intersections have size 1. Formally,

1. $|A_i \cap A_j| = 3$ for all $\{i, j\} \in \binom{[4]}{2}$,
2. $|A_i \cap A_j \cap A_k| = 1$ for all $\{i, j, k\} \in \binom{[4]}{3}$.

Note that there was a typo in the earlier version, as I wrote $|A_i \cap A_j| = 2$. It should be $|A_i \cap A_j| = 3$, as in the preceding text.

Exercise 3.2. Show that if we insist that $|A_i| = 5$ for all i, then the task from the above exercise cannot be solved.

In the spirit of the previous questions, let us call a sequence $(a_1, a_2, \ldots, a_n) \in \mathbb{N}_0$ *feasible* if there are sets $A_1, \ldots, A_n$ such that all k-wise intersections have size a_k. That is, $|A_i| = a_1$ for all i, $|A_i \cap A_j| = a_2$ for all $i \neq j$ and so on. The previous exercise would thus state that $(5, 3, 1)$ is not feasible, but $(6, 3, 1)$ is, as one solution of Exercise 3.1 shows.

This is a typo. The length of the sequence, n, has to be the same as the number of sets. Thus, the previous exercise shows that $(6, 3, 1, 0)$ is feasible and $(5, 3, 1, 0)$ and $(5, 3, 1, 1)$ are infeasible. Sorry for the confusion.

****Exercise 3.3.** Find out something interesting about feasible and infeasible sequences.