Mathematical Foundations of Computer Science

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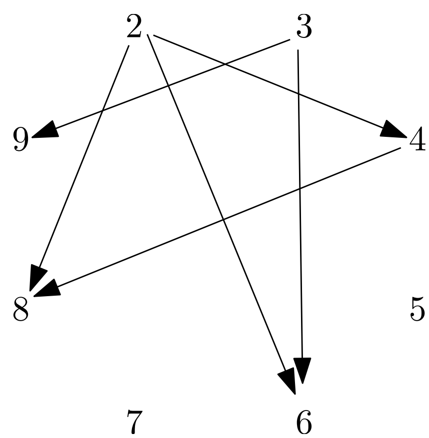
- Homework assignment published on Monday, 2017-02-27
- Work on it and submit questions by Wednesday, 2017-03-01, 18:00 by email to dominik.scheder@gmail.com
- We will discuss some problems on Thursday, 2017-03-02.
- Submit your solution by Sunday, 2017-02-05 to me and TA (kenny_kong@foxmail.com)
- We will give you feedback, you will have an opportunity to revise your solution before we will grade it.

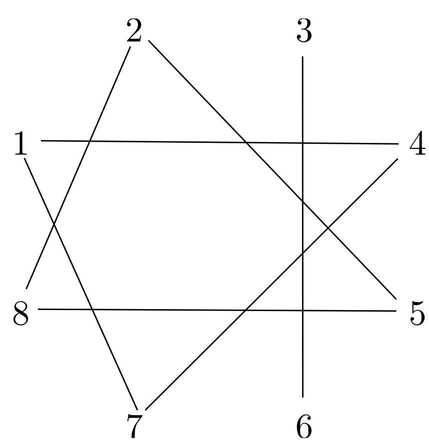
2 Relations

A relation on finite set can be drawn as a diagram, as I illustrate with the relation $\{(a, b) \in \{2, \ldots 9\}^2 \mid a \text{ divides } b\}$.



Note that the relation is reflexive, so I would have to draw a little self-loop at every number. However, exactly because it is reflexive it makes sense to be lazy and drop all these self-loops.

If the relation is symmetric, we do not need the arrowheads. Thus, here is a diagram of the relation $\{(a,b) \in [8]^2 \mid b-a \text{ is divisible by } 3\}$:



Again, I did not draw the self-loops, because I want to keep the picture simple.

Suppose $R \subseteq A \times B$ and $S \subseteq B \times C$ are relations. We define $R \circ S$ to be the relation

$$R \circ S := \{(a,c) \in A \times C \mid \exists b \in B : \ (a,b) \in R, (b,c) \in S\} \ .$$

In the above definition I corrected a typo.

Exercise 2.1. Consider the three following relations on $[8]$:

$R_1 := \{(a,b) \in [8]^2 \mid b-a \text{ is a prime } \}$, $R_2 := \{(a,b) \in [8]^2 \mid a+b \text{ is a square } \}$.

- Draw R_1 and R_2 as a diagram.
- Draw $R_1 \circ R_2$.
- Draw $R_2 \circ R_1$.

Exercise 2.2. Show that $\circ$ is associative. That is, if $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$ are relations, then $(R \circ S) \circ T = R \circ (S \circ T)$. **Remark 1.** Your proof should be quite formal in this case. **Remark 2.** First, think about what exactly you have to prove!

2.1 The Reflexive Transitive Closure

Let $R \subseteq A \times A$ be a relation. We define $R^{(0)} := \{(x, x) \mid x \in A\}$, which is obviously reflexive. For $n \geq 1$, we define inductive $R^{(n)} := R^{(n-1)} \circ R$. Next, we define

$$\tilde{R} := R^{(0)} \cup R^{(1)} \cup R^{(2)} \cup R^{(3)} \cup \cdots = \bigcup_{n \geq 0} R^{(n)} .$$

Exercise 2.3. Let $S := \{(a, b) \in \mathbb{Z}^2 \mid |a - b| = 3\}$ and $T := \{(a, b) \in \mathbb{Z}^2 \mid a - b \text{ is divisible by } 3\}$. Show that $\tilde{S} = T$. Some students apparently missed that the tilde above the S: show that $\tilde{S} = T$, not $S = T$!

Exercise 2.4. Show that $\tilde{R}$ is transitive.

For a relation R we define its reflexive transitive closure R^* by

$$R^* := \bigcap \{S \subseteq A \times A \mid S \supseteq R \text{ and } S \text{ is reflexive and transitive}\} .$$

That is, R^* is the "smallest" relation that (1) contains R, (2) is reflexive, (3) is transitive.

Exercise 2.5. Show that $\tilde{R} = R^*$. **Remark.** Your proof should be very rigorous and quite formal. Think about what exactly you have to prove!

Define $R_k := \{(a, b) \in \mathbb{Z}^2 \mid a - b \text{ is divisible by } k\}$.

Exercise 2.6. Let $a, b \in Z$ be integers. What is $R_a \cap R_b$?

Exercise 2.7. Let $a, b \in Z$ be integers. What is $(R_a \cup R_b)^*$?

3 A String Transformation System

Consider $X = \{0,1\}^*$, the set of all finite 0/1 strings. Let us define two operations on elements of X. First, we can replace two consecutive 0s by a 1, for example $1100 \to 111$. Second, we can insert a 0 between two consecutive 1, for example $011 \to 0101$. We write $x \to y$ if we can obtain y by one of the two operations. Note that $\to$ defines a relation R on X, namely $R := \{(x,y) \in X \times X \mid x \to y\}$.

Exercise 3.1. Let X_3 be the set of 0/1 strings of length at most 3 and let R_3 be the restriction of R to strings in X_3. Formally, $R_3 := R \cap (X_3 \times X_3)$. Draw R_3 as a diagram with arrows.

Consider $\to^*$, the transitive closure of $\to$. That is, $x \to y^*$ if there are $z_1, \ldots, z_n$ such that $x = z_1 \to z_2 \to \cdots \to z_n = y$ (note that for $n = 1$ this includes the case $x = y$). Define

$$\text{Descendant}(x) := \{y \in \{0,1\}^* \mid x \to^* y\} \ ,$$
$$\text{Ancestor}(v) := \{u \in \{0,1\}^* \mid u \to^* v\} \ .$$

Exercise 3.2. Show that |Descendant(x)| is finite for every $x \in \{0,1\}^*$!

Exercise 3.3. Show that |Ancestor(x)| is finite for every $x \in \{0,1\}^*$!