Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

Version D

This homework is special. First of all, it is more difficult than the previous ones. Second, you will have two weeks time to work on them. Third, different groups will get different problem sets. Your solution will be given to two other groups, who give you feedback. Similarly, you will have to read two other groups' solutions and give them feedback.

You have to solve only one version (the one assigned to you), but you have to understand the questions all four versions, since you have to give feedback / grade them, and since all versions might be part of the exam.

Do include your team name in filename of the the pdf file you submit, but NOT in the content of the pdf. The feedback / grading process will be anonymous, i.e., you will not know which group you are reviewing.

- 2017-03-20 (Monday): homework handed out
- 2017-03-22 (Wednesday), 18:00: submit questions
- 2017-03-26 (Sunday), 18:00: submit first solution.
- 2017-03-29 (Wednesday), 18:00: submit your review of the other group's first submission.
- 2017-04-02 (Sunday), 18:00: submit second solution.

- 2017-04-05 (Wednesdy), 18:00: submit your review of the other group's second submission.

- 2017-04-09 (Sunday), 18:00: submit your final solution.

5 A Proof of Dilworth's Theorem

Let $(X, \leq)$ be a partially ordered set. In the video lecture we defined the *width* of X to be the size of a maximal antichain:

$$w(X) := \max\{|A| \mid A \subseteq X \text{ is an antichain}\} .$$

We also talked about the minimum size cp of a chain partition.[1] Formally,

$$cp(X) := \min\{t \mid \exists \text{ chain } C_1, \ldots, C_t \text{ with } C_1 \cup \cdots \cup C_t = X\} .$$

Obviously $w(X) \leq cp(X)$: let A be a maximum size antichain, i.e., $|A| = w(X)$. A chain C_i contains at most one element from A, and all $w(X)$ elements must be covered. Thus, we need at least $|A|$ chains, and $cp(X) \geq w(X)$.

Theorem 5.1 (Dilworth's Theorem)**.** $w(X) = cp(X)$.

In this homework you will prove Dilworth's Theorem.

Exercise 5.2. Show that Dilworth's Theorem follows from the following theorem:

Theorem 5.3. *There exists an antichain $A \subseteq X$ and a partition $X = C_1 \cup \cdots \cup C_t$ into chains such that $t = |A|$.*

Alright, so next you should prove Theorem 5.3. Of course, this is too much for an exercise in an undergraduate course, so let me walk you through a nice inductive proof.

Proof of Theorem 5.3. We proceed by induction on $|X|$. For $|X| = 1$ or $|X| = 0$ the theorem is trivially true.

[1] Note that we can equivalently talk about chain *covers*. The chains need not be disjoint; in fact, if they are not, we can safely remove elements from certain chains to make them disjoint, thus convert a cover into a partition.

Exercise 5.4. As a warm-up, show that Dilworth's Theorem holds for $|X| = 2$ and $|X| = 3$ by drawing some nice pictures that take care of all different order relations on X.

Alright, so let $|X| = n$ and let's assume Theorem 5.3 holds for all ordered sets on at most $n - 1$ elements. We set $w := w(X)$, the width of X. Our goal is to partition X into w chains.

Now let's get going. Let C be a maximum size *chain* in X (that's true; a chain, not an antichain). Note that $X \setminus C$ is again an ordered set.

Exercise 5.5. Show how to finish the proof in the case that $w(X \setminus C) \leq w-1$.

Now that this case is taken care of, let's try to handle the case that $w(X \setminus C) = w$. That means, suppose there is an antichain $A \subseteq X \setminus C$ of size $|A| = w$.

Exercise 5.6. Show that this case can happen! That is, draw some partial ordered set (i.e., draw its Hasse diagram), highlight some maximum size chain C in it, such that $w(X \setminus C) = w(X)$.

Now let us fix this antichain A and define the following sets:

$$X^+ := \{x \in X \mid \exists\, a \in A \text{ with } a < x\} \ ,$$
$$X^- := \{x \in X \mid \exists\, a \in A \text{ with } x < a\} \ .$$

Recall that we write $x < y$ as a shorthand for "$x \leq y$ and $x \neq y$. Let us write $C = \{c_1, c_2, \ldots, c_h\}$ with $h = |C|$ and $c_1 < c_2 < \cdots < c_h$ (remember that C is of maximum size and therefore $h = h(X)$, the *height* of X). We list some basic facts:

Proposition 5.7. *The following things are true about C, A, X^+, and X^-:*

1. $X = X^- \cup A \cup X^+$,
2. $c_1 \in X^-$,
3. $c_h \in X^+$.

Exercise 5.8. Draw a picture that nicely illustrates Proposition 5.7.

Exercise 5.9. Prove Proposition 5.7.

Exercise 5.10. Show that $w(X^- \cup A) = w(X^+ \cup A) = |A|$.

By the above proposition, $|X^- \cup A| \leq n-1$ since this set does not contain c_h. Thus, we can apply the inductive hypothesis to $X^- \cup A$ and obtain a chain partition of $X^- \cup A$ into w chains:

$$X^- \cup A = C_1^- \cup \cdots \cup C_w^- \ .$$

By the analogous argument, we see that $|X^+ \cup A|$, apply induction and find a chain partition of $X^+ \cup A$ into w chains:

$$X^+ \cup A = C_1^+ \cup \cdots \cup C_w^+ \ .$$

Exercise 5.11. Finish the proof of Theorem 5.3.

$\square$