上 海 交 通 大 学 试 卷（A卷）

（2016 至 2017 学年 第二学期）

班级号_____ 学号_____ 姓名____

课程名称 __Mathematical Foundations of Computer Science (CS499)_____ 成绩____

Each problem is worth 20 points. 100 points will give you full grade. Please note: (1) Justify your answers. (2) The problems are not sorted from easiest to hardest. So check first before wasting time on the hardest problem. Also, all problems have some easy sub-problems.

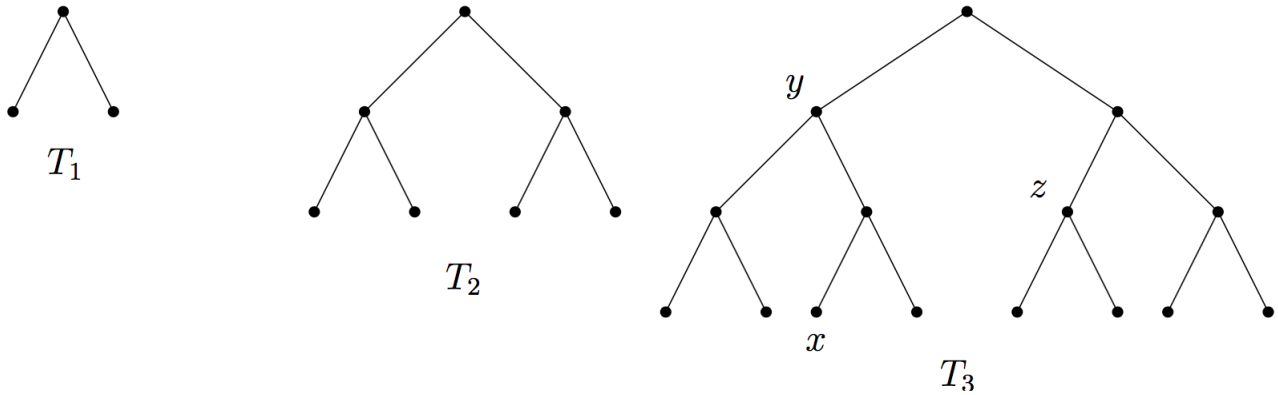
Problem 1

Recall Lucas' Theorem, which states that $\binom{n}{k}$ is odd if $k \preceq n$ and even otherwise, where $k \preceq n$ is the "bitwise comparison" of k and n. For example, $4 \preceq 12$ since 4 in binary is 100 and 12 is 1100, and bit-by-bit we have $0100 \leq 1100$. To the contrary, $0010 \not\leq 1100$ and therefore $2 \not\preceq 12$. Lucas' Theorem now implies that $\binom{12}{4}$ is odd and $\binom{12}{2}$ is even. A fact that can be easily verified by hand. Next, you are going to use Lucas' Theorem to derive a bunch of rules:

1. For which k is $\binom{2k+1}{k}$ odd? Find a rule and prove it!
2. For which k is $\binom{4k}{k}$ odd? Find a rule and prove it!
3. For which k is $\binom{2k}{k}$ odd? Prove this fact WITHOUT using Lucas' Theorem!

Problem 2

Let T_n *binary tree of depth* n. Instead of giving a formal definition, let me draw the first few T_n:



Let V_n denote the set of vertices of T_n. We define an ordering on V_n: $x \preceq y$ if x is an ancestor of y in T_n. For example, in T_3 above we have $y \preceq x$ but $x \not\preceq z$ and $z \not\preceq x$.

1. An ordering is a relation and thus a subset of $V_n \times V_n$. What is $|\preceq|$? That is, how many pairs (x, y) are there with $x \preceq y$? Give a closed formula and prove it.

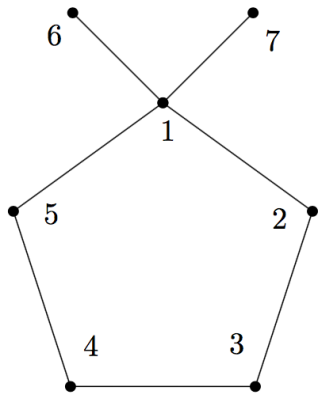
2. What is the largest antichain of $(V_n, \preceq)$? Prove you answer! For this, you may use Dilworth's Theorem, if you want!

3. Obviously, $\preceq$ is *not* a linear ordering (for $n \geq 1$). Find and describe some *linear extension* of $\preceq$. Try to find a very simple one.

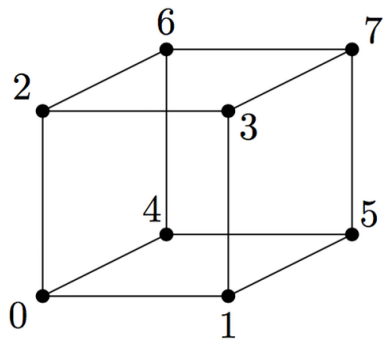
4. Find two linear orderings $\leq_1$ and $\leq_2$ such that $\preceq = \leq_1 \cap \leq_2$. That is, for all x, y in the tree, $x \preceq y$ if and only if $x \leq_1 y$ and $y \leq_1 x$.

Problem 3

Recall that an automorphism of a graph $G = (V, E)$ is a bijection $\pi : V \to V$ that is an isomorphism. That is, $\{u, v\} \in E$ if and only if $\{\pi(u), \pi(v)\} \in E$. Clearly, the identity function $\pi(u) = u$ is always an automorphism. We call a graph *asymmetric* if the identity is the only automorphism. Consider the following graph:



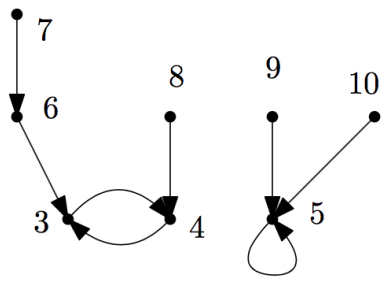
1. How many automorphisms does it have?
2. Which edge can you add to this graph to make it asymmetric? Justify your answer, i.e., argue why the resulting graph is asymmetric.
3. Consider the three-dimensional Hamming cube:



What is the smallest number of edges you have to remove to make this graph asymmetric? That is, find the smallest such k, exhibit a set of k edges such that removing these edges makes the graph asymmetric, and argue why removing $j < k$ edges is not enough.

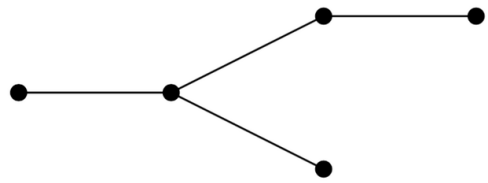
Problem 4

Recall the definition of the *core* of a function. For example, the core of the function below is $\{1, 2, 3\}$.



1. Let $s_3(n)$ be the number of functions $f : [n] \to [n]$ with core size 3. Give a summation formula for $s_3(n)$. Don't try to give a *closed* formula. You will have to use $\sum$, $\binom{j}{i}$ and so on.

2. For a graph G, let $P_k(G)$ denote the number of paths of length k in G. For example, $P_1(G)$ is simply the number of edges, and for this graph



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we have $P_2(G) = 4$, $P_3(G) = 2$, and $P_4(G) = 0$. Now consider $\mathbb{E}[P_2(T)]$, the expectation of $P_2(T)$ when T is a random tree on the vertex set $\{1, \ldots, n\}$. Formally,

$$\mathbb{E}[P_2(T)] = \frac{1}{n^{n-2}} \sum_T P_2(T) \ .$$

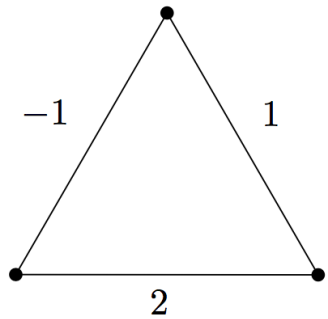
Give a closed formula for this in terms of $s_3(n)$. By this I mean, your formula must not contain any $\sum$, but may contain $s_3(n)$.

3. For a graph G, let $w_2(G) := \sum_v (d(v))^2$, the sum of the squares of the degrees. Give a formula for $w_2(G)$ in terms of $P_2(T)$.

Problem 5

1. Give an example of a sequence $(d_1, \ldots, d_n)$, $n \geq 4$, which is the score of a multigraph but not of a graph.

2. Consider graphs with integer edge weights. These are graphs $G = (V, E)$ together with an edge weight function $w : E \to \mathbf{Z}$. Yes, weights can be negative integers here.



A graph with integer weights and score $0, 1, 3$.

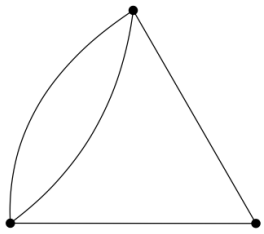
State an integer weights graph score theorem. That is, something like

Theorem. *A sequence $d_1, \ldots, d_n$ is an integer weights graph score if and only if* `<put a very simple criterion here>`.

3. Prove your theorem.

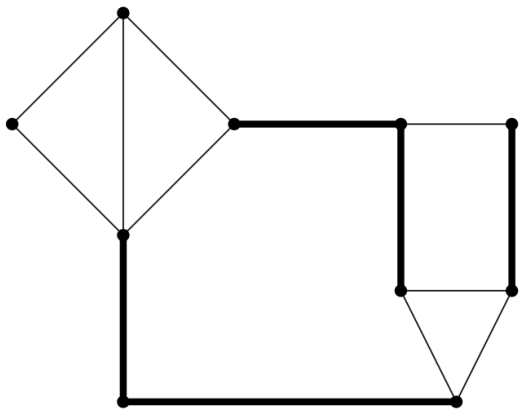
Problem 6

Recall that a multigraph is a graph that can have parallel edges. For a multigraph G, we denote by $t(G)$ the number of spanning trees of G. Furthermore, recall that C_k is a cycle with k vertices (and k edges). Suppose we take C_k, remove one edge and replace it by a parallel edges. We call the resulting graph $C_{k,a}$.



The graph $C_{3,2}$, a triangle with one edge replaced by two parallel edges. Note that $t(C_{3,2}) = 5$.

1. Find an explicit formula for $t(C_{k,a})$ and argue why it holds.
2. Look at the following graph G:



   Thin edges have weight 1, thick edges have weight 2. Draw a minimum spanning tree of G.

3. How many minimum spanning trees does G have?
4. Draw a multigraph H (without edge weights) that has as many spanning trees as G has *minimum* spanning trees.

题号	1	2	3	4	5	6				
得分										
批阅人(流水阅卷教师签名处)										

我承诺，我将严格遵守考试纪律。

承诺人：