Design and Analysis of Algorithms (XIII)

Linear Programming: Applications

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Shortest Path

Shortest Path

Shortest path problem gives a weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbb{Q}^+$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the weight of a shortest path from s to t.

Shortest Path in LP

$$\max d_t$$

$$\begin{aligned} d_v &\le d_u + w(u,v) && (u,v) \in E \\ d_s &= 0 && \\ d_i &\ge 0 && i \in V \end{aligned}$$

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Q: Another formalization?

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Let $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$; that is, $\mathcal{S}$ is the set of all s-t cuts in the graph. Then we can model the shortest s-t path problem with the following integer program,

$$\min \sum_{e \in E} w_e x_e$$

$$\begin{aligned} \sum_{e \in \delta(S)} x_e \geq 1 \quad & S \in \mathcal{S} \\ x_e \in \{0, 1\} \quad & e \in E \end{aligned}$$

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- Can we relax the restriction $x_e \in \{0,1\}$ to $0 \leq x_e \leq 1$?
- How about $x_e \geq 0$?

Shortest Path in LP

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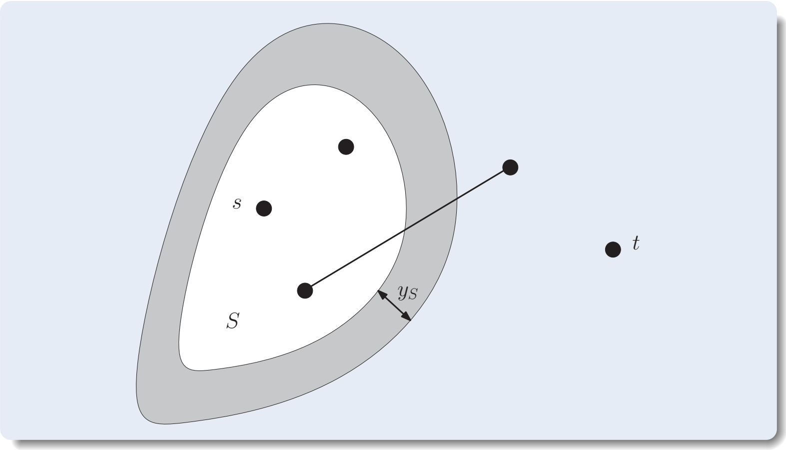
$$\max \sum_{S \in \mathcal{S}} y_S$$

$$\sum_{S \in \mathcal{S}, e \in \delta(S)} y_S \leq w_e \quad e \in E$$

$$y_S \geq 0 \quad S \in \mathcal{S}$$

The Moat



Max-Flow Min-Cut in LP

Shipping Oil

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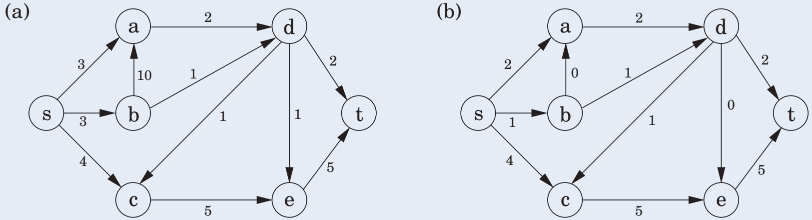
The goal is to ship as much oil as possible from the source to the sink.

Each pipeline has a maximum capacity it can handle, and there are no opportunities for storing oil en route.

A Flow Example



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The networks consist of a directed graph $G = (V, E)$; two special nodes $s, t \in V$, a source and sink of G; and capacities $c_e > 0$ on the edges.

Aim to send as much oil as possible from s to t without exceeding the capacities of any of the edges.

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1. It doesn't violate edge capacities: $0 \le f_e \le c_e$ for all $e \in E$.
2. For all nodes u except s and t, the amount of flow entering u equals the amount leaving

$$\sum_{(w,v)\in E} f_{wu} = \sum_{(u,z)\in E} f_{uz}$$

In other words, flow is conserved.

Maximizing Flow

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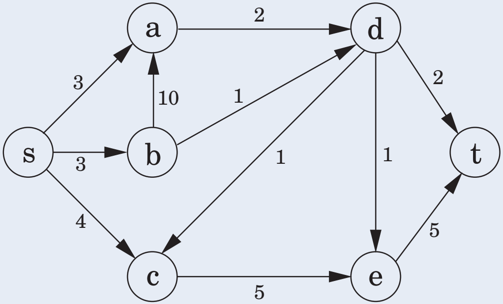
This is a linear program. The maximum-flow problem reduces to linear programming.

The Example



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27 constraints:

- 11 for nonnegativity (such as $f_{sa} \geq 0$),
- 11 for capacity (such as $f_{sa} \leq 3$),
- 5 for flow conservation (one for each node of the graph other than s and t, such as $f_{sc} + f_{dc} = f_{ce}$).

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The advantage of making this modification is that we can now require flow conservation at s and t as well.

Another Representation

$$
\begin{aligned}
& \max f_{ts} \\
& f_{ij} \leq c_{ij} && (i,j) \in E \\
& \sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 && i \in V \\
& f_{ij} \geq 0 && (i,j) \in E
\end{aligned}
$$

Remark

Ford-Fulkersons algorithm can be regarded as a special algorithm of linear programs.

Min-Max Relations and Duality

LP for Max Flow

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- p_i: potentials on nodes.

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This solution defines an $s-t$ cut $(X, \overline{X})$, where X is the set of potential 1 nodes, and $\overline{X}$ the set of potential 0 nodes.

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The objective function value is precisely the capacity of the cut $(X, \overline{X})$, and hence $(X, \overline{X})$ must be a minimum $s - t$ cut.

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We will say that this program is the LP relaxation of the integer program.

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The constraint matrix of this program is totally unimodular, Thus, the dual program always has an integral optimal solution.

More Examples

Set Cover

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- Input: A set of elements U, sets $S_1, \ldots, S_m \subseteq U$
- Output: A selection of the S_i whose union is U.
- Cost: Number of sets picked.

Set Cover

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$$
\begin{aligned}
\min \quad & \sum_{S \in \mathcal{S}} x_S & \\
& \sum_{S: e \in S} x_S \geq 1, & e \in U \\
& x_S \geq 0, & S \in \mathcal{S}
\end{aligned}
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Set Cover

- Input: A set of elements U, sets $S_1, \dots, S_m \subseteq U$, and a cost function $c : \mathcal{S} \to \mathbb{Q}^+$.
- Output: A selection of the S_i whose union is U.
- Cost: Sum of costs of set picked.

The special case, in which all subsets are of unit cost, will be called the cardinality set cover problem.

Set Cover

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$$
\begin{aligned}
\min \quad & \sum_{S \in \mathcal{S}} c(S) x_S \\
& \sum_{S : e \in S} x_S \geq 1, && e \in U \\
& x_S \geq 0, && S \in \mathcal{S}
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\max \quad & \sum_{e \in U} y_e \\
& \sum_{e : e \in S} y_e \leq c(S), && S \in \mathcal{S} \\
& y_e \geq 0, && e \in U
\end{aligned}
$$

Quiz: Set Multicover

Each element, e, needs to be covered a specified integer number, r_e, of times.

The objective again is to cover all elements up to their coverage requirements at minimum cost.

Each set can be picked at most once.

Integer Program

Let $r_e \in \mathbb{Z}_+$ be the coverage requirement for each element $e \in U$.

$$\begin{aligned} \min \quad & \sum_{S \in \mathcal{S}} c(S) x_S \\ & \sum_{S : e \in S} x_S \geq r_e, && e \in U \\ & x_S \in \{0, 1\}, && S \in \mathcal{S} \end{aligned}$$

Linear Program Relaxation

In the LP-relaxation, the constraints $x_S \leq 1$ are no longer redundant.

$$
\begin{aligned}
\min \quad & \sum_{S \in \mathcal{S}} c(S) x_S & \\
& \sum_{S : e \in S} x_S \geq r_e, & e \in U \\
& -x_S \geq -1, & S \in \mathcal{S} \\
& x_S \geq 0, & S \in \mathcal{S}
\end{aligned}
$$

Dual Program

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The additional constraints in the primal lead to new variables, z_S, in the dual.

$$
\begin{aligned}
\max \quad & \sum_{e \in U} r_e y_e - \sum_{S \in \mathcal{S}} z_S & \\
& \Big(\sum_{e: e \in S} y_e\Big) - z_S \leq c(S), & S \in \mathcal{S} \\
& y_e \geq 0, & e \in U \\
& z_S \geq 0, & S \in \mathcal{S}
\end{aligned}
$$

Referred Materials

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Content of this lecture comes from Section 12.2 in [Vaz04] and Section 7.3 in [WS11].

Suggest to read Section 26.1 and 26.2 in [CLRS09].